and does not depend upon \( \theta \), as is seen by the transformation \( w_i = x_i/\theta \), \( i = 1, 2, \ldots, n \). So \( Y \) and \( Z \) are independent statistics.

This special case of the independence of \( Y \) and \( Z \) concerning one sufficient statistic \( Y \) and one parameter \( \theta \) was first observed by Hogg (1953) and then generalized to several sufficient statistics for more than one parameter by Basu (1955) and is usually called Basu's theorem.

Due to these results, sufficient statistics are extremely important and estimation problems are based upon them when they exist.

**Exercises**

6.7.1. Let \( X_1, X_2, \ldots, X_n \) be a random sample from \( N(0, \sigma^2) \).

(a) Find a sufficient statistic \( Y \) for \( \sigma^2 \).

(b) Show that the maximum likelihood estimator for \( \sigma^2 \) is a function of \( Y \).

(c) Is the maximum likelihood estimator for \( \sigma^2 \) unbiased?

6.7.2. Let \( X_1, X_2, \ldots, X_n \) be a random sample from a Poisson distribution with mean \( \lambda > 0 \). Find the conditional probability \( P(X_1 = x_1, \ldots, X_n = x_n | Y = y) \), where \( Y = X_1 + \cdots + X_n \) and the nonnegative integers \( x_1, x_2, \ldots, x_n \) sum to \( y \). Note that this probability does not depend on \( \lambda \).

6.7.3. Write the bivariate normal pdf \( f(x, y; \theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \) in exponential form and show that \( Z_1 = \sum_{i=1}^n X_i^2 \), \( Z_2 = \sum_{i=1}^n Y_i \), \( Z_3 = \sum_{i=1}^n X_i Y_i \), \( Z_4 = \sum_{i=1}^n X_i^2 \), and \( Z_5 = \sum_{i=1}^n Y_i \) are joint sufficient statistics for \( \theta_1, \theta_2, \theta_3, \theta_4, \) and \( \theta_5 \).

6.7.4. Let \( X_1, X_2, \ldots, X_n \) be a random sample from a distribution with pdf \( f(x; \theta) = \theta x^{\theta-1}, 0 < x < 1 \), where \( 0 < \theta \).

(a) Find a sufficient statistic \( Y \) for \( \theta \).

(b) Show that the maximum likelihood estimator \( \hat{\theta} \) is a function of \( Y \).

(c) Argue that \( \hat{\theta} \) is also sufficient for \( \theta \).

6.7.5. Let \( X_1, X_2, \ldots, X_n \) be a random sample from a gamma distribution with \( \alpha = 1 \) and \( 1/\theta > 0 \). Show that \( Y = \sum_{i=1}^n X_i \) is a sufficient statistic, and \( Y \) has a gamma distribution with parameters \( n \) and \( 1/\theta \), and \( (n-1)/Y \) is an unbiased estimator of \( \theta \).

6.7.6. Let \( X_1, X_2, \ldots, X_n \) be a random sample from a gamma distribution with known parameter \( \alpha \) and unknown parameter \( \theta > 0 \).

(a) Show that \( Y = \sum_{i=1}^n X_i \) is a sufficient statistic for \( \theta \).

(b) Show that the maximum likelihood estimator of \( \theta \) is a function of \( Y \) and is an unbiased estimator of \( \theta \).

6.7.7. Let \( X_1, X_2, \ldots, X_n \) be a random sample from the distribution with pmf \( f(x; p) = p(1-p)^{x-1}, x = 1, 2, 3, \ldots \), where \( 0 < p \leq 1 \).

(a) Show that \( Y = \sum_{i=1}^n X_i \) is a sufficient statistic for \( p \).

(b) Find a function of \( Y \) that is an unbiased estimator of \( \theta = 1/p \).

6.7.8. Let \( X_1, X_2, \ldots, X_n \) be a random sample from \( N(0, \sigma^2) \), where \( \sigma^2 = \theta > 0 \) is unknown. Argue that the sufficient statistic \( Y = \sum_{i=1}^n X_i^2 \) for \( \theta \) and \( Z = \sum_{i=1}^n X_i/\sum_{i=1}^n X_i \) are independent. Hint: Let \( X_i = \theta \omega_i, i = 1, 2, \ldots, n \), in the multivariate integral representing \( E[e^{\sum_{i=1}^n Z_i}] \).

6.7.9. Let \( X_1, X_2, \ldots, X_n \) be a random sample from \( N(\theta_1, \theta_2) \). Show that the sufficient statistics \( Y_1 = \bar{X} \) and \( Y_2 = S^2 \) are independent of the statistic \( Z = \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2 / S^2 \) because \( Z \) has a distribution that is free of \( \theta_1 \) and \( \theta_2 \).

Hint: Let \( w_i = (x_i - \theta_1)/\sqrt{\theta_2} \), \( i = 1, 2, \ldots, n \), in the multivariate integral representing \( E[e^{\sum_{i=1}^n Z_i}] \).

6.7.10. Find a sufficient statistic for \( \theta \), given a random sample, \( X_1, X_2, \ldots, X_n \), from a distribution with pdf

\[ f(x; \theta) = \frac{1}{\Gamma(\theta)} \theta^{\frac{1}{\theta}} x^{\frac{1}{\theta}-1} (1-x)^{\frac{1}{\theta}-1}, 0 < x < 1 \]

6.7.11. Let \( X_1, X_2, \ldots, X_n \) be a random sample from a distribution with pdf \( f(x; \theta) = (1/2)\theta^2 x^{\frac{1}{\theta}} e^{-\theta x}, 0 < x < \infty \). Show that \( Y = \sum_{i=1}^n X_i \) and \( Z = (X_1 + X_2)/Y \) are independent.

6.7.12. Let \( X_1, X_2, \ldots, X_n \) be a random sample from \( N(0, \sigma^2) \), where \( n \) is odd. Let \( Y \) and \( Z \) be the mean and median of the sample. Argue that \( Y \) and \( Z - Y \) are independent so that the variance of \( Z \) is \( \text{Var}(Y) + \text{Var}(Z - Y) \).

We know that \( \text{Var}(Y) = \sigma^2/n \), so that we could estimate the variance of \( Z \) by Monte Carlo. This might be more efficient than estimating \( \text{Var}(Z) \) directly since \( \text{Var}(Z - Y) \leq \text{Var}(Z) \). This scheme is often called the Monte Carlo Swindle.