Chapter 6 Point Estimation

Exercises

6.6.1. Let \( X_1, X_2, \ldots, X_n \) be a random sample from \( N(\theta, \sigma^2) \), where \( \sigma^2 \) is known.

(a) Show that \( Y = (X_1 + X_2)/2 \) is an unbiased estimator of \( \theta \).

(b) Find the Rao–Cramér lower bound for the variance of an unbiased estimator of \( \theta \) for a general \( n \).

(c) What is the efficiency of \( Y \) in part (a)?

6.6.2. Let \( X_1, X_2, \ldots, X_n \) denote a random sample from \( b(1, p) \). We know that \( \bar{X} \) is an unbiased estimator of \( p \) and that \( \text{Var}(\bar{X}) = p(1 - p)/n \). (See Exercise 6.4-12.)

(a) Find the Rao–Cramér lower bound for the variance of every unbiased estimator of \( p \).

(b) What is the efficiency of \( \bar{X} \) as an estimator of \( p \)?

6.6.3. (Continuation of Exercise 6.4-2.) In sampling from a normal distribution with known mean \( \mu \), the maximum likelihood estimator of \( \sigma^2 \) is \( \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_i - \mu)^2}{n} \).

(a) Determine the Rao–Cramér lower bound.

(b) What is the approximate distribution of \( \hat{\sigma}/\theta \), where \( \theta = \sigma^2 \)?

6.6.4. Find the Rao–Cramér lower bound, and thus the asymptotic variance of the maximum likelihood estimator \( \hat{\theta} \), if the random sample \( X_1, X_2, \ldots, X_n \) is taken from each of the distributions having the following pdfs:

(a) \( f(x; \theta) = \left(\frac{1}{\theta^2}\right) x e^{-x/\theta} \), \( 0 < x < \infty \), \( 0 < \theta < \infty \).

(b) \( f(x; \theta) = \left(\frac{1}{2\theta^3}\right) x^2 e^{-x/\theta} \), \( 0 < x < \infty \), \( 0 < \theta < \infty \).

(c) \( f(x; \theta) = \left(\frac{1}{\theta}\right) x^{(1-\theta)/\theta} \), \( 0 < x < 1 \), \( 0 < \theta < \infty \).

6.7 SUFFICIENT STATISTICS

We first define a sufficient statistic \( Y = u(X_1, X_2, \ldots, X_n) \) for a parameter, using a statement that, in most books, is given as a necessary and sufficient condition for sufficiency, namely, the well-known Fisher–Neyman factorization theorem. We do this because we find that readers at the introductory level can apply such a definition easily. However, using this definition, we shall note, by examples, its implications, one of which is also sometimes used as the definition of sufficiency. An understanding of Example 6.7-3 is most important in an appreciation of the value of sufficient statistics.

**Definition 6.7-1**

**Factorization Theorem** Let \( X_1, X_2, \ldots, X_n \) denote random variables with joint pdf or pmf \( f(x_1, x_2, \ldots, x_n; \theta) \), which depends on the parameter \( \theta \). The statistic \( Y = u(X_1, X_2, \ldots, X_n) \) is sufficient for \( \theta \) if and only if

\[
f(x_1, x_2, \ldots, x_n; \theta) = \phi[u(x_1, x_2, \ldots, x_n); \theta] h(x_1, x_2, \ldots, x_n),
\]

where \( \phi \) depends on \( x_1, x_2, \ldots, x_n \) only through \( u(x_1, \ldots, x_n) \) and \( h(x_1, \ldots, x_n) \) does not depend on \( \theta \).

Let us consider several important examples and consequences of this definition. We first note, however, that in all instances in this book the random variables \( X_1, X_2, \ldots, X_n \) will be of a random sample, and hence their joint pdf or pmf will be of the form

\[
f(x_1; \theta)f(x_2; \theta) \cdots f(x_n; \theta).
\]

**Example 6.7-1** Let \( X_1, X_2, \ldots, X_n \) denote a random sample from a Poisson distribution with parameter \( \lambda > 0 \). Then