AMC 8169
30 May 2003 Disk used
ARTICLE IN PRESS
No. of Pages 13, DTD=4.3.1
SPS, Chennai

ELSEVIER
Applied Mathematics and Computation xxx (2003) xxx–xxx
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Oscillation criteria for impulsive parabolic boundary value problem with delay

Xilin Fu [a], LieJune Shiau [b,*]

[a] *Department of Mathematics, Shandong Normal University, Jinan, Shandong 250014, PR China*

[b] *Department of Mathematics, University of Houston, Clear Lake, 2700 Bay Area Blvd., Houston, TX 77058, USA*

Abstract

In this work, we study the criteria of oscillatory solutions to impulsive parabolic boundary value problem with delay. First, we consider two types of boundary condition which resolve in oscillatory solutions in the impulsive problem with delay, then we further reduce the oscillation criteria for the problem.

Keywords: Oscillation criteria; Impulsive; Parabolic; Delay; Eventually positive solution

1. Introduction

In 1991, Erbe et al. [1] first studied impulsive parabolic equations in application models. Later in 1994 Bainov et al. [2], extended the impulsive study in hyperbolic partial differential equations for periodic boundary value problem. Fu and Liu [3] then further studied oscillation criteria for impulsive hyperbolic problems in 1997. In this work, we study oscillation criteria for impulsive parabolic problem with delay which was not previously studied.

* Corresponding author.
E-mail address: shiau@cl.uh.edu (L. Shiau).

doi:10.1016/S0096-3003(03)00659-3

AMC 8169
30 May 2003 Disk used
ARTICLE IN PRESS
No. of Pages 13, DTD = 4.3.1
SPS, Chennai

2. Preliminaries

Consider the following impulsive parabolic system with delay

$$\begin{aligned} &u_t = a(t)\Delta u + b(t)\Delta u(t-\rho,x) - p(t,x)f(u(t-\sigma,x)), \quad t \neq t_k, \\ &u(t_k^+,x) - u(t_k^-,x) = I(t,x,u), \quad t = t_k, \;\; k = 1,2,\ldots, \end{aligned} \tag{2.1}$$

where

1. Δ is the Laplacian in $\mathbf{R}^n$; $u = u(t,x)$ for $(t,x) \in G = \mathbf{R}_+ \times \Omega$, where Ω is a bounded domain in $\mathbf{R}^n$ with a smooth boundary $\partial\Omega$ and $\mathbf{R}_+ = [0,+\infty)$,
2. $0 < t_1 < t_2 < \cdots < t_k < \cdots$, where $\lim_{k\to\infty} t_k = +\infty$,
3. $a(t),\ b(t) \in PC[\mathbf{R}_+,\mathbf{R}_+],\ p(t,x) \in PC[\mathbf{R}_+ \times \overline{\Omega},\mathbf{R}_+]$, and PC is the class of piecewise continuous functions in t with discontinuities of first kind only at $t = t_k$, $k = 1,2,\ldots$ and left continuous at $t = t_k$ and $\overline{\Omega}$ is the closure of Ω; Also, we have $f(u) \in C[\mathbf{R},\mathbf{R}]$,
4. both ρ and σ are positive constants, and
5. $I : \mathbf{R}_+ \times \overline{\Omega} \times \mathbf{R} \mapsto \mathbf{R}$.

We shall consider two kinds of boundary condition in this study:

$$\frac{\partial u}{\partial N} + h(x)u = g(t,x), \quad (t,x) \in \mathbf{R}_+ \times \partial\Omega, \;\; t \neq t_k \tag{2.2}$$

and

$$u = \varphi(t,x), \quad (t,x) \in \mathbf{R}_+ \times \partial\Omega, \;\; t \neq t_k, \tag{2.3}$$

where $h(x) \in (\partial\Omega,(0,+\infty))$, $g(t,x)$ and $\varphi(t,x) \in PC[\mathbf{R}_+ \times \partial\Omega, R]$, and N is the unit out normal vector to $\partial\Omega$.

We know that the solutions $u(t,x)$ of problem (2.1) with boundary condition either (2.2) or (2.3) are both piecewise continuous functions with points of discontinuity of first kind at $t = t_k$, $k = 1,2,\ldots$ Following the convention, we shall assume that they are left continuous. That is, at the moments of impulse, the following relations $u(t_k^-,x) = u(t_k,x)$ and $u(t_k^+,x) = u(t_k,x) + I(t_k,x,u(t_k,x))$ are satisfied. Next, we recall the definition of oscillatory solutions.

Definition 2.1. A nonzero solution $u(t,x)$ of boundary value problem (2.1), (2.2) or problem (2.1), (2.3) is said to be nonoscillatory in the domain G, if there exists a number $\tau \geqslant 0$ such that $u(t,x)$ has a constant sign for $(t,x) \in [\tau,+\infty) \times \Omega$. Otherwise, it is said to be oscillatory.

Now, we are ready to develop oscillation criteria.

AMC 8169
30 May 2003 Disk used
ARTICLE IN PRESS
No. of Pages 13, DTD = 4.3.1
SPS, Chennai

3. Sufficient conditions for oscillatory solutions

Consider the following Robin eigenvalue problem

$$\begin{aligned}&\Delta u + \lambda u = 0, \quad x \in \Omega,\\ &\frac{\partial u}{\partial N} + h(x)u = 0, \quad x \in \partial\Omega,\end{aligned} \tag{3.1}$$

where λ is a constant. Then, we have the following properties which will lead to the oscillation.

Lemma 3.1. *If $h(x) \in C(\partial\Omega, (0, +\infty))$, then the Robin eigenvalue problem* (3.1) *has a minimum positive eigenvalue λ_0 and the corresponding eigenfunction $\eta(t)$ is positive on Ω (see Theorem* 3.3.22 *of* [4]).

Lemma 3.2. *Let $h(x) \in C(\partial\Omega, (0, +\infty))$ and the following assumptions*

(A1) *$f(u)$ is a positive and convex function in $\mathbf{R}_+$,*
(A2) *for any function $u \in PC[\mathbf{R}_+ \times \overline{\Omega}, \mathbf{R}_+]$ and constants $\alpha_k > 0$ such that $\int_\Omega I(t_k, x, u(t_k, x))\,\mathrm{d}x \leqslant \alpha_k \int_\Omega u(t_k, x)\,\mathrm{d}x$, $k = 1, 2, \ldots$*

also hold. If $u(t, x)$ is a positive solution of problem (2.1), (2.2) *in the domain $[\tau, +\infty) \times \Omega$ for some $\tau \geqslant 0$, then the impulsive differential inequality with delay*

$$\begin{aligned}&U'(t) + \lambda_0 a(t)U(t) + \lambda_0 b(t)U(t-\rho) + P(t)f(U(t-\sigma)) \leqslant R(t), \quad t \neq t_k,\\ &U(t_k^+) \leqslant (1+\alpha_k)U(t_k), \quad k = 1, 2, \ldots\end{aligned} \tag{3.2}$$

has the eventually positive solution

$$U(t) = \frac{1}{\int_\Omega \eta(x)\,\mathrm{d}x}\int_\Omega u(t, x)\eta(x)\,\mathrm{d}x, \tag{3.3}$$

where

$$P(t) = \min_{x\in\overline{\Omega}}\{p(t, x)\} \quad \text{and}$$

$$R(t) = \frac{1}{\int_\Omega \eta(x)\,\mathrm{d}x}\left[a(t)\int_{\partial\Omega} \eta(x)g(t, x)\,\mathrm{d}S + b(t)\int_{\partial\Omega}\eta(x)g(t-\rho, x)\,\mathrm{d}S\right],$$
$$t \neq t_k, \ \mathrm{d}S \text{ is an area element of } \partial\Omega.$$

Proof. Let $u(t, x)$ be a positive solution of problem (2.1), (2.2) in the domain $[\tau, +\infty) \times \Omega$ for some $\tau \geqslant 0$. For $t \neq t_k$, there exists a $t^* \geqslant \tau$ such that $u(t-\rho, x) > 0$, $u(t-\sigma, x) > 0$ for $(t, x) \in [t^*, +\infty) \times \Omega$. Multiplying both sides of (2.1) by the eigenfunction $\eta(x)$ and integrating with respect to x over the domain Ω, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t,x)\eta(x)\,\mathrm{d}x &\\ = a(t)\int_\Omega \Delta u(t,x)\eta(x)\,\mathrm{d}x + b(t)\int_\Omega \Delta u(t-\rho,x)\eta(x)\,\mathrm{d}x &\\ - \int_\Omega p(t,x)f(u(t-\sigma,x))\eta(x)\,\mathrm{d}x, \quad t\neq t_k,\ t\geqslant t^*. &\end{aligned} \tag{3.4}$$

From (A1) and Jensen's inequality, it follows that

$$\begin{aligned}\int_\Omega p(t,x)f(u(t-\sigma,x))\eta(x)\,\mathrm{d}x &\\ \geqslant P(t)\int_\Omega \eta(x)\,\mathrm{d}x\cdot f\left(\frac{1}{\int_\Omega \eta(x)\,\mathrm{d}x}\int_\Omega u(t-\sigma,x)\eta(x)\,\mathrm{d}x\right), \quad t\neq t_k,\ t\geqslant \tau. &\end{aligned} \tag{3.5}$$

Using Green's Theorem and Lemma 3.1, we have

$$\begin{aligned}\int_\Omega \Delta u(t,x)\eta(x)\,\mathrm{d}x &= \int_{\partial\Omega}\left(\eta\frac{\partial u}{\partial N} - u\frac{\partial \eta}{\partial N}\right)\mathrm{d}S + \int_\Omega u\Delta\eta\,\mathrm{d}x\\ &= \int_{\partial\Omega}(\eta(g-hu) - u(-h\eta))\,\mathrm{d}S + \int_\Omega u(-\lambda_0\eta)\,\mathrm{d}x\\ &= \int_{\partial\Omega}\eta g\,\mathrm{d}S - \lambda_0\int_\Omega u\eta\,\mathrm{d}x, \quad t\neq t_k,\ t\geqslant t^*\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\int_\Omega \Delta u(t-\rho,x)\eta(x)\,\mathrm{d}x &\\ = \int_{\partial\Omega}\eta(x)g(t-\rho,x)\,\mathrm{d}S - \lambda_0\int_\Omega u(t-\rho,x)\eta(x)\,\mathrm{d}x, \quad t\neq t_k,\ t\geqslant \tau. &\end{aligned} \tag{3.7}$$

Combining (3.4)–(3.7), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t,x)\eta(x)\,\mathrm{d}x + \lambda_0 a(t)\int_\Omega u(t,x)\eta(x)\,\mathrm{d}x + \lambda_0 b(t)\int_\Omega u(t-\rho,x)\eta(x)\,\mathrm{d}x &\\ + \int_\Omega \eta(x)\,\mathrm{d}x\cdot P(t)\cdot f\left(\frac{1}{\int_\Omega \eta(x)\,\mathrm{d}x}\int_\Omega u(t-\sigma,x)\eta(x)\,\mathrm{d}x\right) &\\ \leqslant a(t)\int_{\partial\Omega}\eta(x)g(t,x)\,\mathrm{d}S + b(t)\int_{\partial\Omega}\eta(x)g(t-\rho,x)\,\mathrm{d}S, \quad t\neq t_k,\ t\geqslant t^*. &\end{aligned} \tag{3.8}$$

For $t = t_k$, using (A2) we have

$$\begin{aligned}&\int_\Omega (u(t_k^+, x) - u(t_k, x))\eta(x)\,\mathrm{d}x\\&\quad = \int_\Omega I(t_k, x, u(t_k, x))\eta(x)\,\mathrm{d}x \leqslant \alpha_k \int_\Omega u(t_k, x)\eta(x)\,\mathrm{d}x, \quad k = 1, 2, \ldots\end{aligned}$$

That is

$$\int_\Omega u(t_k^+, x)\eta(x)\,\mathrm{d}x \leqslant (1 + \alpha_k) \int_\Omega u(t_k, x)\eta(x)\,\mathrm{d}x, \quad k = 1, 2, \ldots \tag{3.9}$$

Thus, inequalities (3.8) and (3.9) imply that the function $U(t)$ defined by (3.3) is a positive solution of the impulsive differential inequality with delay in (3.2) for $t \geqslant t^*$. The proof of Lemma 3.2 is therefore completed. □

Theorem 3.3. *Assume that conditions* (A1) *and* (A2) *hold, and* $h \in C(\partial\Omega, (0, +\infty))$. *If we further assume that*

$$\begin{aligned}\text{(A3)}\quad & f(-u) = -f(u) \quad \textit{for } u \in (0, +\infty),\\ & I(t_k, x, -u(t_k, x)) = -I(t_k, x, u(t_k, x)), \quad k = 1, 2, \ldots\end{aligned}$$

and the impulsive differential inequality with delay in both problems (3.2) *and*

$$\begin{aligned}&U'(t) + \lambda_0 a(t)U(t) + \lambda_0 b(t)U(t - \rho) + P(t)f(U(t - \sigma)) \leqslant -R(t), \quad t \neq t_k,\\&U(t_k^+) \leqslant (1 + \alpha_k)U(t_k), \quad k = 1, 2, \ldots\end{aligned} \tag{3.10}$$

have no eventually positive solutions, then each nonzero solution of the problem (2.1), (2.2) *is oscillatory in the domain* G.

Proof. Assuming the contrary is true. Let $u(t, x)$ be a nonzero solution of the problem (2.1), (2.2) which has a constant sign in the domain $[\tau, +\infty) \times \Omega$ for some $\tau \geqslant 0$. We first consider the case of $u(t, x) > 0$ for $(t, x) \in [\tau, +\infty) \times \Omega$. From Lemma 3.2, it follows that the function $U(t)$ defined by (3.3) is an eventually positive solution of the inequality (3.2), which contradicts the condition of the theorem. If $u(t, x) < 0$ for $(t, x) \in [\tau, +\infty) \times \Omega$, then the function

$$\tilde{u}(t, x) = -u(t, x), \quad (t, x) \in [\tau, +\infty) \times \Omega$$

is a positive solution of the following impulsive parabolic boundary value problem with delay

$$
\begin{aligned}
&u_t = a(t)\Delta u + b(t)\Delta u(t-\rho,x) - p(t,x)f(u(t-\sigma,x)), \quad t \neq t_k,\ (t,x)\in G\\
&\frac{\partial u}{\partial N} + h(x)u = -g(t,x), \quad t\neq t_k,\ (t,x)\in R_+\times\partial\Omega\\
&u(t_k^+,x) - u(t_k^-,x) = I(t,x,u), \quad t = t_k,\ k = 1,2,\ldots
\end{aligned}
\tag{3.11}
$$

and satisfies

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tilde u(t,x)\eta(x)\,\mathrm{d}x + \lambda_0 a(t)\int_\Omega \tilde u(t,x)\eta(x)\,\mathrm{d}x + \lambda_0 b(t)\int_\Omega \tilde u(t-\rho,x)\eta(x)\,\mathrm{d}x\\
&\quad + \int_\Omega \eta(x)\,\mathrm{d}x\cdot P(t)\cdot f\left(\frac{1}{\int_\Omega \eta(x)\,\mathrm{d}x}\int_\Omega \tilde u(t-\sigma,x)\eta(x)\,\mathrm{d}x\right)\\
&\leqslant -\left(a(t)\int_{\partial\Omega}\eta(x)g(t,x)\,\mathrm{d}S + b(t)\int_{\partial\Omega}\eta(x)g(t-\rho,x)\,\mathrm{d}S\right), \quad t\neq t_k,\ t\geqslant\tau
\end{aligned}
\tag{3.12}
$$

and

$$
\int_\Omega \tilde u(t_k^+,x)\eta(x)\,\mathrm{d}x \leqslant (1+\alpha_k)\int_\Omega \tilde u(t_k,x)\eta(x)\,\mathrm{d}x, \quad k=1,2,\ldots
$$

Thus it follows that the function

$$
\widetilde U(t) = \frac{1}{\int_\Omega \eta(x)\,\mathrm{d}x}\int_\Omega \tilde u(t,x)\eta(x)\,\mathrm{d}x
$$

is a positive solution of the inequality (3.10) for $t \geqslant t^* \geqslant \tau$ which also contradicts the conditions of the theorem. This completes the proof of Theorem 3.3. □

Now, if we set $g\equiv 0$ in the proof of Theorem 3.3, then we can also obtain the following theorem.

Theorem 3.4. *Assume that conditions* (A1)–(A3) *hold, and* $h\in C(\partial\Omega,(0,+\infty))$. *If the impulsive differential inequality with delay*

$$
\begin{aligned}
&U'(t) + \lambda_0 a(t)U(t) + \lambda_0 b(t)U(t-\rho) + P(t)f(U(t-\sigma)) \leqslant 0, \quad t\neq t_k,\\
&U(t_k^+) \leqslant (1+\alpha_k)U(t_k), \quad k=1,2,\ldots
\end{aligned}
\tag{3.13}
$$

has no eventually positive solutions, then each nonzero solution of the system (2.1), *satisfying the boundary condition*

$$
\frac{\partial u}{\partial N} + h(x)u = 0, \quad (t,x)\in R_+\times\partial\Omega,\ t\neq t_k \tag{3.14}
$$

AMC 8169
30 May 2003 Disk used
ARTICLE IN PRESS
No. of Pages 13, DTD = 4.3.1
SPS, Chennai

is oscillatory in the domain G.

The following fact shall be used later in the proof of Lemma 3.5. Consider the Dirichlet problem

$$\begin{aligned}&\Delta u + \lambda u = 0, \quad x \in \Omega,\\ &u = 0, \quad x \in \partial\Omega,\end{aligned}$$

where $\lambda =$ constant. It is well known that the smallest eigenvalue λ^* and the corresponding eigenfunction $\Phi(x)$ are positive.

Lemma 3.5. *Assume that* (A1) *and* (A2) *hold. If $u(t,x)$ is a positive solution of the problem* (2.1), (2.3) *in the domain $[\tau, +\infty] \times \Omega$ for some $\tau \geqslant 0$, then the impulsive differential inequality with delay*

$$\begin{aligned}&V'(t) + \lambda^* a(t)V(t) + \lambda^* b(t)V(t-\rho) + p(t)f(V(t-\sigma)) \leqslant Q(t), \quad t \neq t_k,\\ &V(t_k^+) \leqslant (1+\alpha_k)V(t_k), \quad k = 1,2,\ldots\end{aligned} \tag{3.15}$$

has the eventually positive solution

$$V(t) = \frac{1}{\int_\Omega \Phi(x)\,\mathrm{d}x}\int_\Omega u(t,x)\Phi(x)\,\mathrm{d}x, \tag{3.16}$$

where

$$Q(t) = -\frac{1}{\int_\Omega \Phi(x)\,\mathrm{d}x}\left[a(t)\int_{\partial\Omega}\varphi(t,x)\frac{\partial\Phi}{\partial N}\,\mathrm{d}S + b(t)\int_{\partial\Omega}\varphi(t-\rho,x)\frac{\partial\Phi}{\partial N}\,\mathrm{d}S\right],$$
$$t \neq t_k.$$

Proof. Let $u(t,x)$ be a positive solution of the problem (2.1), (2.3) in the domain $[\tau, +\infty] \times \Omega$ for some $\tau \geqslant 0$. For $\tau \neq t_k$, there exist a $t^* \geqslant \tau$ such that $u(t-\rho,x), u(t-\sigma,x) > 0$ for $(t,x) \in [t^*, +\infty) \times \Omega$. Multiplying both sides of (2.1) by the eigenfunction $\Phi(x)$ and integrating with respect to x over the domain Ω, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t,x)\Phi(x)\,\mathrm{d}x\\ &\quad = a(t)\int_\Omega \Delta u(t,x)\Phi(x)\,\mathrm{d}x + b(t)\int_\Omega \Delta u(t-\rho,x)\Phi(x)\,\mathrm{d}x\\ &\qquad - \int_\Omega p(t,x)f(u(t-\sigma,x))\Phi(x)\,\mathrm{d}x, \quad t \neq t_k,\ t \geqslant t^*.\end{aligned} \tag{3.17}$$

From (A1) and Jensen's inequality, it follows that

$$\int_\Omega p(t,x) f(u(t-\sigma,x))\Phi(x)\,\mathrm{d}x \geqslant P(t)\int_\Omega \Phi(x)\,\mathrm{d}x \cdot f\left(\frac{1}{\int_\Omega \Phi(x)\,\mathrm{d}x}\int_\Omega u(t-\sigma,x)\Phi(x)\,\mathrm{d}x\right), \quad t \neq t_k,\ t \geqslant t^*. \tag{3.18}$$

Using Green's Theorem, we have

$$\int_\Omega \Delta u(t,x)\Phi(x)\,\mathrm{d}x = \int_\Omega \left(\Phi\frac{\partial u}{\partial N} - u\frac{\partial \Phi}{\partial N}\right)\mathrm{d}S + \int_\Omega u\Delta\Phi(x)\,\mathrm{d}x = \int_{\partial\Omega}\left(-\varphi(t,x)\frac{\partial\Phi}{\partial N}\right)\mathrm{d}S + \int_\Omega u(-\lambda^*\Phi)\,\mathrm{d}x, \quad t \neq t_k,\ t \geqslant t^*. \tag{3.19}$$

And

$$\int_\Omega \Delta u(t-\rho,x)\Phi(x)\,\mathrm{d}x = -\int_{\partial\Omega}\varphi(t-\rho,x)\frac{\partial\Phi}{\partial N}\,\mathrm{d}S - \lambda^*\int_\Omega u(t-\rho,x)\Phi(x)\,\mathrm{d}x, \quad t \neq t_k,\ t \geqslant t^*. \tag{3.20}$$

Combining (3.17)–(3.20), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t,x)\Phi(x)\,\mathrm{d}x &+ \lambda^* a(t)\int_\Omega u(t,x)\Phi(x)\,\mathrm{d}x + \lambda^* b(t)\int_\Omega u(t-\rho,x)\Phi(x)\,\mathrm{d}x \\ &+ \int_\Omega \Phi(x)\,\mathrm{d}x\cdot P(t)\cdot f\left(\frac{1}{\int_\Omega\Phi(x)\,\mathrm{d}x}\int_\Omega u(t-\sigma,x)\Phi(x)\,\mathrm{d}x\right) \\ &\leqslant -a(t)\int_{\partial\Omega}\varphi(t,x)\frac{\partial\Phi}{\partial N}\,\mathrm{d}S - b(t)\int_{\partial\Omega}\varphi(t-\rho,x)\frac{\partial\Phi}{\partial N}\,\mathrm{d}S, \quad t \neq t_k,\ t \geqslant t^*.\end{aligned} \tag{3.21}$$

For $t = t_k$, using (A2) we have

$$\int_\Omega u(t_k^+,x)\Phi(x)\,\mathrm{d}x \leqslant (1+\alpha_k)\int_\Omega u(t_k,x)\Phi(x)\,\mathrm{d}x, \quad k = 1,2,\ldots \tag{3.22}$$

Thus we can see that the function $V(t)$ defined in (3.16) is a positive solution of the impulsive differential inequality with delay (3.15) for $t \geqslant t^*$. Thus the proof of Lemma 3.5 is complete. □

AMC 8169
30 May 2003 Disk used
ARTICLE IN PRESS
No. of Pages 13, DTD = 4.3.1
SPS, Chennai

Theorem 3.6. *Assume that conditions* (A1)–(A3) *hold. If we assume further that both the impulsive differential inequality with delay* (3.15) *and the impulsive differential inequality with delay*

$$\begin{aligned}&V'(t)+\lambda^* a(t)V(t)+\lambda^* b(t)V(t-\rho)+P(t)f(V(t-\sigma))\leqslant -Q(t), \quad t\neq t_k,\\ &V(t_k^+)\leqslant (1+\alpha_k)V(t_k), \quad k=1,2,\ldots\end{aligned} \tag{3.23}$$

have no eventually positive solutions, then each nonzero solution of the problem (2.1), (2.3) *is oscillatory in the domain G.*

Since this proof is similar to Theorem 3.3, we omit it. Furthermore, if we set $\varphi\equiv 0$, then we can have the following theorem.

Theorem 3.7. *Assume that conditions* (A1)–(A3) *hold. If the impulsive differential inequality with delay*

$$\begin{aligned}&V'(t)+\lambda^* a(t)V(t)+\lambda^* b(t)V(t-\rho)+P(t)f(V(t-\sigma))\leqslant 0, \quad t\neq t_k,\\ &V(t_k^+)\leqslant (1+\alpha_k)V(t_k), \quad k=1,2,\ldots\end{aligned} \tag{3.24}$$

have no eventually positive solutions, then each nonzero solution of system (2.1) *satisfying the boundary condition*

$$u=0, \quad (t,x)\in R_t\times\partial\Omega, \quad t\neq t_k \tag{3.25}$$

is oscillatory in the domain G.

4. Further oscillation criteria

From the discussion in previous section, it follows that the problem of establishing oscillation criteria for the impulsive parabolic system (2.1) satisfying some boundary condition can be reduced to the investigation of the properties of the solutions of the first order impulsive differential inequalities. In this section, we shall establish some further oscillation criteria for the impulsive parabolic systems.

Lemma 4.1. *If there exists a constant* $\delta>0$ *such that*

$$t_{k+1}-t_k\geqslant\delta, \quad k=1,2,\ldots,$$

then there exists a constant $r\in N$ *such that the number of the impulse moments in each of the intervals* $[t,t+\rho^*]$, $t>0$ *is not greater than r, where* $\rho^*=max\{\rho,\sigma\}$.

Proof. It is easy to see that in each interval of the form $[t, t+\rho^*]$, $t > 0$, we have at most $1 + [\frac{\rho^*}{\delta}]$ impulse moments. Thus we can choose

$$r \geqslant 1 + \left[\frac{\rho^*}{\delta}\right]. \qquad \square$$

Theorem 4.2. *Assume that conditions* (A1)–(A3) *hold, and* $h \in C(\partial\Omega, (0, +\infty))$. *If we assume further that*

1. *there exists a constant* $\delta > 0$, *such that*

$$t_{k+1} - t_k \geqslant \delta, \quad k = 1, 2, \ldots,$$

2. *there exists a constant* $\alpha > 0$, *such that*

$$0 < \alpha_k < \alpha, \quad k = 1, 2, \ldots,$$

3.

$$\limsup_{k\to+\infty} \int_{t_k}^{t_k+\rho} b(s) e^{\lambda_0 \int_{s-\rho}^{s} a(\xi)\,d\xi} ds > \frac{1}{\lambda_0}(1+\alpha)^{2r},$$

then each nonzero solution of the problem (2.1), (3.14) *is oscillatory in the domain* G.

Proof. Let $u(t,x)$ be a nonzero solution of the problem (2.1), (3.14) which has a constant sign in the domain $[\tau, +\infty] \times \Omega$ for some $\tau \geqslant 0$. If $u(t,x) > 0$ for $(t,x) \in [\tau, +\infty] \times \Omega$, then we can see that the function $U(t)$ defined by (3.3) a positive solution of the inequality (3.15) for $t \geqslant \tau + \rho^*$ and $U(t-\rho) > 0$, $f(U(t-\sigma)) > 0$ for $t \geqslant \tau + \rho^*$. For $t \neq t_k$, from (3.15) we get

$$U'(t) + \lambda_0 a(t) U(t) + \lambda_0 b(t) U(t-\rho) \leqslant 0, \quad t \geqslant \tau + \rho^*. \tag{4.1}$$

Multiply (4.1) by $e^{\lambda_0 \int_T^t a(\xi)\,d\xi}$ for $t > T \geqslant \tau + \rho^*$, and set

$$y(t) = U(t) e^{\lambda_0 \int_T^t a(\xi)\,d\xi}, \quad t > T. \tag{4.2}$$

We obtain

$$y'(t) + e^{\lambda_0 \int_T^t a(\xi)\,d\xi} \lambda_0 b(t) y(t-\rho) e^{-\lambda_0 \int_T^{t-\rho} a(\xi)\,d\xi} \leqslant 0, \quad t \neq t_k, \;\; t > T + \rho. \tag{4.3}$$

From (4.2) and (4.3), it follows that $y(t)$ is a nonincreasing function. For $t = t_k$,

$$\begin{aligned} \Delta y(t_k) = y(t_k^+) - y(t_k) &= [U(t_k^+) - U(t_k)] e^{\lambda_0 \int_T^t a(\xi)\,d\xi} \\ &\leqslant \alpha_k U(t_k) e^{\lambda_0 \int_T^t a(\xi)\,d\xi} = \alpha_k y(t_k). \end{aligned}$$

Integrate (4.3) from t_k to $t_k+\rho$ and use Lemma 4.1, we have

$$y(t_k+\rho) - y(t_k^+) - \sum_{s=k}^{k+r-1} \alpha_s y(t_s) + \int_{t_k}^{t_k+\rho} \lambda_0 b(s) \mathrm{e}^{\lambda_0 \int_{s-\rho}^{s} a(\xi)\,\mathrm{d}\xi} y(s-\rho)\,\mathrm{d}s \leqslant 0. \tag{4.4}$$

Note that

$$y(s-\rho) \geqslant \frac{y(s-\rho)}{(1+\alpha)^r}. \tag{4.5}$$

From (4.4) and (4.5), we have

$$\begin{aligned} &\frac{\lambda_0}{(1+\alpha)^r} \int_{t_k}^{t_k+\rho} b(s) \mathrm{e}^{\lambda_0 \int_{s-\rho}^{s} a(\xi)\,\mathrm{d}\xi} y(s-\rho)\,\mathrm{d}s \\ &\quad \leqslant y(t_k^+) - y(t_k+\rho) + \sum_{s=k}^{k+r-1} \alpha_s y(t_s) \leqslant (1+\alpha_k) y(t_k) + \sum_{s=k+1}^{k+r-1} \alpha_s y(t_s) \end{aligned}$$

and

$$\frac{\lambda_0}{(1+\alpha)^r} y(t_k) \int_{t_k}^{t_k+\rho} b(s) \mathrm{e}^{\lambda_0 \int_{s-\rho}^{s} a(\xi)\,\mathrm{d}\xi} \mathrm{d}s \leqslant (1+\alpha) y(t_k) + \alpha \sum_{s=k+1}^{k+r-1} y(t_s). \tag{4.6}$$

But

$$\begin{aligned} &y(t_{k+1}) \leqslant y(t_k^+) \leqslant (1+\alpha_k) y(t_k) \leqslant (1+\alpha) y(t_k), \\ &y(t_{k+2}) \leqslant y(t_{k+1}^+) \leqslant (1+\alpha_{k+1}) y(t_{k+1}) \leqslant (1+\alpha) y(t_{k+1}) \leqslant (1+\alpha)^2 y(t_k), \\ &\ldots \\ &y(t_{k+r-1}) \leqslant \cdots \leqslant (1+\alpha)^{r-1} y(t_k). \end{aligned}$$

Then

$$\sum_{s=k+1}^{k+r-1} y(t_s) \leqslant y(t_k) \sum_{i=1}^{r-1} (1+\alpha)^i = y(t_k)(1+\alpha) \frac{(1+\alpha)^{r-1}-1}{\alpha}. \tag{4.7}$$

From (4.6) and (4.7), it follows that

$$\begin{aligned} &\frac{\lambda_0}{(1+\alpha)^r} y(t_k) \int_{t_k}^{t_k+\rho} b(s) \mathrm{e}^{\lambda_0 \int_{s-\rho}^{s} a(\xi)\,\mathrm{d}\xi} \mathrm{d}s \\ &\quad \leqslant (1+\alpha) y(t_k) + \alpha y(t_k)(1+\alpha) \frac{(1+\alpha)^{r-1}-1}{\alpha} = y(t_k)(1+\alpha)^r. \end{aligned}$$

That is

AMC 8169
30 May 2003 Disk used
ARTICLE IN PRESS
No. of Pages 13, DTD = 4.3.1
SPS, Chennai

$$\int_{t_k}^{t_k+\rho} b(s)\mathrm{e}^{\lambda_0 \int_{s-\rho}^{s} a(\xi)\,\mathrm{d}\xi}\mathrm{d}s \leqslant \frac{1}{\lambda_0}(1+\alpha)^{2r}.$$

The last inequality contradicts condition 3 in Theorem 4.2. If $u(t,x) < 0$ for $(t,x) \in [\tau, +\infty) \times \Omega$, then it is easy to check that $-u(t,x)$ is a positive solution of the problem (2.1) and (3.14) for $(t,x) \in [\tau, +\infty) \times \Omega$. Thus there is contradiction, by the analogous arguments, the proof is therefore completed. □

It is important to note that the resulting condition involving the coefficient of delayed Laplacian $b(t)$. This result is obtained through the method of Robin eigenfunction. But this result cannot be obtained by the method in [3]. We can prove the following result by the analogous arguments as in the proof of Theorem 4.2.

Theorem 4.3. *Assume that conditions* (A1)–(A3) *hold, and* $h \in C(\partial\Omega, (0, +\infty))$. *If we assume further that*

1. $\frac{f(u)}{u} \geqslant A$, $u \in (0, +\infty)$ *for some constant* $A > 0$,
2. *there exists a constant* $\delta > 0$, *such that*

$$t_{k+1} - t_k \geqslant \delta, \quad k = 1, 2, \ldots,$$

3. *there exists a constant* $\alpha > 0$, *such that*

$$0 < \alpha_k < \alpha, \quad k = 1, 2, \ldots,$$

4.

$$\limsup_{k\to+\infty} \int_{t_k}^{t_k+\sigma} P(s)\mathrm{e}^{\lambda_0 \int_{s-\sigma}^{s} a(\xi)\,\mathrm{d}\xi}\mathrm{d}s > \frac{1}{A}(1+\alpha)^{2r},$$

then each nonzero solution of the problem (2.1), (3.14) *is oscillatory in the domain G.*

Theorem 4.4. *Assume that conditions* (A1)–(A3) *hold. If we assume further that conditions* 1 *and* 2 *in Theorem* 4.2 *and* 3

$$\limsup_{k\to+\infty} \int_{t_k}^{t_k+\rho} b(s)\mathrm{e}^{\lambda^* \int_{s-\rho}^{s} a(\xi)\,\mathrm{d}\xi}\mathrm{d}s > \frac{1}{\lambda^*}(1+\alpha)^{2r}$$

also hold, then each nonzero solution of the problem (2.1), (3.25) *is oscillatory in the domain G.*

Theorem 4.5. *Assume that conditions* (A1)–(A3) *hold. If we assume further that conditions* 1–3 *in Theorem* 4.3 *and* 4

$$\limsup_{k\to+\infty}\int_{t_k}^{t_k+\sigma} P(s)\mathrm{e}^{\lambda^*\int_{s-\sigma}^{s} a(\xi)\,\mathrm{d}\xi}\,\mathrm{d}s > \frac{1}{A}(1+\alpha)^{2r}$$

also hold, then each nonzero solution of the problem (2.1), (3.25) *is oscillatory in the domain G.*

Acknowledgements

Xilin Fu acknowledges the support by National Natural Science Foundation of China (10171057) and by Natural Science Foundation of Shandong Province. LieJune Shiau acknowledges the partial support by Faculty Research Support Fund of University of Houston, Clear Lake.

References

[1] L.H. Erbe, J.I. Freedman, X.Z. Liu, J.J. Wu, Comparison principles for impulsive parabolic equations with applications to models of single species growth, J. Aust. Math. Soc. Sec. B 32 (1991) 382–400.
[2] D. Bainov, Z. Kamont, E. Minchev, Periodic boundary value problem for impulsive hyperbolic partial differential equations of first order, Appl. Math. Comput. 80 (1994) 1–10.
[3] X. Fu, X. Liu, Oscillation criteria for impulsive hyperbolic systems, Dyn. Cont. Disc. Imp. Syst. 3 (1997) 225–244, 3139–3161.
[4] Q.X. Ye, Z.Y. Li, Introduction to Reaction Diffusion Equations, Science Press, Beijing, 1990.