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The Numerical Simulation of Friction Constrained Motions (I): One Degree of Freedom Models

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Abstract—In a previous article [1], the authors discussed the time-discretization of those relations modeling a class of dynamical systems with friction. The main goal of this article is to address similar problems using a more sophisticated friction model giving a better description of the system behavior when the velocities are close to zero. These investigations are motivated by the need for more accurate friction models in the software simulating the motion of mechanical systems, such as the remote manipulators of the Space Shuttle or of the International Space Station. As a first step, we shall consider one degree of freedom systems. However, the methods discussed in this article can be easily generalized to higher number of degrees of freedom elasto-dynamical systems; these generalizations will be the object of another publication. The content can be summarized as follows. We first discuss several models of the constrained motions under consideration, including a rigorous formulation involving a kind of dynamical multiplier. Next, in order to treat friction, we introduce an implicit/explicit numerical scheme which is unconditionally stable, and easy to implement and generalize to more complicated systems. Indeed, the above scheme can be coupled, via operator-splitting, to schemes classically used to solve differential equations from frictionless elasto-dynamics. The above schemes are validated through numerical experiments. © 2004 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Motivated by the real time simulation of *elasto-dynamical systems* with *friction*, we introduced in [1], a family of numerical schemes taking advantage of the existence of a *friction multiplier*.

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Discrepancies between simulations and real life results lead engineers to refine their friction models in order to improve simulation quality, particularly at very low (relative) velocities, i.e., when friction forces dominate the dynamics of the system under consideration. For simplicity, we shall consider *one degree of freedom* systems only, but the methodology discussed here extends easily to higher number of degrees of freedom.

2. MODELING OF FRICTION CONSTRAINED MOTIONS: SPLITTING OF THE MODEL

Some remote manipulator system simulators use multidimensional generalization of the *one degree of freedom* model below to describe *dry friction constrained motions*:

$$m\ddot{x} + kx + c(\mathrm{sgn}(\dot{x}) - \gamma(\dot{x})) = f, \qquad \text{on } (0,T), \quad \text{with } x(0) = x_0, \quad \dot{x}(0) = v_0, \tag{1}$$

where in (1), x is a displacement (here $x(t) \in \mathbf{R}$), m is a *mass*, k is a *stiffness coefficient*, c is a *friction coefficient*, f is an *external force*, $T \in (0, +\infty]$, and γ is a *nondecreasing Lipschitz continuous function*, vanishing at 0 and such that $\lim_{\xi\to\pm\infty} \gamma(\xi) = \pm\beta$, with $0 < \beta < 1$.

Remark 1. The case $\gamma = 0$ has been discussed in, e.g., [1,2].

Typical functions γ are provided by

$$\gamma(\xi) = \frac{\beta\xi}{\sqrt{\epsilon^2 + \xi^2}} \tag{2}$$

or

$$\gamma(\xi) = \frac{\beta\xi}{\epsilon}, \quad \text{if } |\xi| \le \epsilon, \qquad \gamma(\xi) = \beta\,\mathrm{sgn}(\xi), \quad \text{if } |\xi| \ge \epsilon. \tag{3}$$

Operator γ has been introduced to take into consideration the following well-known fact: when there is dry friction, the force necessary to put the system into motion, starting from rest, is higher than the one necessary to maintain the motion.

A "rigorous" equivalent formulation of (1) is given by

$$\begin{gathered} m\ddot{x} + kx + c\lambda - c\gamma(\dot{x}) = f \quad \text{and} \quad \lambda\dot{x} = |\dot{x}|, \qquad |\lambda| \le 1, \\ \text{on } (0,T), \quad \text{with } x(0) = x_0, \quad \dot{x}(0) = v_0. \end{gathered} \tag{4}$$

In (4), the *multiplier* λ models the dry friction forces. Proving the existence of a pair $\{x, \lambda\}$ verifying (4) is easy; inspired by [3] (see also [2]), we approximate (4) (and equation (1)) by

$$m\ddot{x}_\eta + kx_\eta + \frac{c\dot{x}_\eta}{\sqrt{\eta^2 + \dot{x}_\eta^2}} - c\gamma(\dot{x}_\eta) = f, \qquad \text{on } (0,T), \quad \text{with } x_\eta(0) = x_0, \quad \dot{x}_\eta(0) = v_0, \tag{5}$$

with η positive, and denote $\dot{x}_\eta/\sqrt{\eta^2 + \dot{x}_\eta^2}$ by λ_η. Suppose that $0 < T < +\infty$ and $f \in L^\infty(0,T)$. Problem (5) has clearly a unique solution and, using *Ascoli's theorem*, we can prove that

$$\lim_{\eta\to 0}\{x_\eta, \lambda_\eta\} = \{x, \lambda\}, \qquad \text{in } \left(W^{2,\infty}(0,T) \times L^\infty(0,T)\right) \text{ weak-*}, \tag{6}$$

where $\{x, \lambda\}$ is a solution (necessarily unique) of problem (4). Relation (6) implies that $\lim_{\eta\to 0} x_\eta = x$ in $C^1[0,T]$. In order to *decouple* the numerical treatment of the *elasticity* and *friction* operators (kx and $c(\mathrm{sgn}(\dot{x}) - \gamma(\dot{x}))$, respectively, here), we observe that systems (1) and (4) are *equivalent* to

$$\begin{aligned} &m\dot{v} + c(\mathrm{sgn}(v) - \gamma(v)) + kx = f, && \text{on } (0,T), \\ &\dot{x} = v, && \text{on } (0,T), \end{aligned} \tag{7}$$

$$x(0) = x_0, \qquad v(0) = v_0,$$

and

$$\begin{aligned} &m\dot{v} + c\lambda - c\gamma(v) + kx = f, && \text{on } (0,T),\\ &\dot{x} = v, && \text{on } (0,T),\\ &\lambda v = |v|, \quad |\lambda| \le 1, && \text{on } (0,T),\\ &x(0) = x_0, \qquad v(0) = v_0, \end{aligned} \tag{8}$$

respectively. Let N be a positive integer and $\Delta t = T/N$, we denote $n\Delta t$ by t^n. Among the many possible *operator-splitting schemes* available to time-discretize (7) and (8), we advocate the one below, particularly easy to implement and generalize to higher dimensions (we consider the discretization of (7) only, the application to (8) being an obvious variant):

$$x^0 = x_0, \qquad v^0 = v_0, \tag{9}$$

for $n = 1, \ldots, N, x^n$ and v^n being known, solve

$$\begin{aligned} &m\dot{v} + c(\mathrm{sgn}(v) - \gamma(v)) = f, && \text{on } \left(t^n, t^{n+1}\right),\\ &\dot{x} = 0, && \text{on } \left(t^n, t^{n+1}\right),\\ &v(t^n) = v^n, \qquad x(t^n) = x^n; \qquad v^{n+1/2} = v\left(t^{n+1}\right), \qquad x^{n+1/2} = x^n, \end{aligned} \tag{10}$$

$$\begin{aligned} &m\dot{v} + kx = 0, && \text{on } \left(t^n, t^{n+1}\right),\\ &\dot{x} = v, && \text{on } \left(t^n, t^{n+1}\right),\\ &v(t^n) = v^{n+1/2}, \qquad x(t^n) = x^{n+1/2}; \qquad v^{n+1} = v\left(t^{n+1}\right), \qquad x^{n+1} = x\left(t^{n+1}\right). \end{aligned} \tag{11}$$

Problem (11) is *equivalent* to

$$\begin{aligned} &m\ddot{x} + kx = 0, && \text{on } \left(t^n, t^{n+1}\right),\\ &x(t^n) = x^{n+1/2}, \qquad \dot{x}(t^n) = v^{n+1/2}; \qquad x^{n+1} = x\left(t^{n+1}\right), \qquad v^{n+1} = \dot{x}\left(t^{n+1}\right). \end{aligned} \tag{12}$$

The numerical solution of the subinitial value problems (10) and (11),(12) will be discussed in Sections 3 and 4, respectively.

Remark 2. A symmetrized (in the sense of [4]) variant of scheme (9)–(11) reads as follows (with $t^{n+1/2} = (n + 1/2)\Delta t$):

$$x^0 = x_0, \qquad v^0 = v_0, \tag{13}$$

for $n = 1, \ldots, N$, x^n and v^n being known, solve

$$\begin{aligned} &m\dot{v} + c(\mathrm{sgn}(v) - \gamma(v)) = f, && \text{on } \left(t^n, t^{n+1/2}\right),\\ &\dot{x} = 0, && \text{on } \left(t^n, t^{n+1/2}\right),\\ &v(t^n) = v^n, \qquad x(t^n) = x^n; \qquad v^{n+1/2} = v\left(t^{n+1/2}\right), \qquad x^{n+1/2} = x^n, \end{aligned} \tag{14}$$

$$\begin{aligned} &m\dot{v} + kx = 0, && \text{on } (0, \Delta t),\\ &\dot{x} = v, && \text{on } (0, \Delta t),\\ &v(0) = v^{n+1/2}, \qquad x(0) = x^{n+1/2}; \qquad \hat{v}^{n+1/2} = v(\Delta t), \qquad \hat{x}^{n+1/2} = x(\Delta t), \end{aligned} \tag{15}$$

$$\begin{aligned} &m\dot{v} + c(\mathrm{sgn}(v) - \gamma(v)) = f, && \text{on } \left(t^{n+1/2}, t^{n+1}\right),\\ &\dot{x} = 0, && \text{on } \left(t^{n+1/2}, t^{n+1}\right),\\ &v\left(t^{n+1/2}\right) = \hat{v}^{n+1/2}, \quad x\left(t^{n+1/2}\right) = \hat{x}^{n+1/2}; \quad v^{n+1} = v\left(t^{n+1}\right), \quad x^{n+1} = \hat{x}^{n+1/2}. \end{aligned} \tag{16}$$

3. NUMERICAL SOLUTION OF THE SUBPROBLEMS OF TYPE (10)

Problem (10) is a special case of

$$\begin{aligned} &m\dot{w} + c(\mathrm{sgn}(w) - \gamma(w)) = f, \qquad \text{on } (t_0, t_f), \\ &w(t_0) = w_0, \end{aligned} \tag{17}$$

(with $t_0 < t_f$) itself equivalent to

$$\begin{aligned} &m\dot{w} + c\lambda - c\gamma(w) = f, \qquad \text{on } (t_0, t_f), \\ &\lambda w = |w|, \quad |\lambda| \le 1, \qquad \text{on } (t_0, t_f), \\ &w(t_0) = w_0. \end{aligned} \tag{18}$$

Suppose that $f \in L^\infty(t_0, t_f)$; then problem (17) (respectively, (18)) has a unique solution in $W^{1,\infty}(t_0, t_f)$ (respectively, $W^{1,\infty}(t_0, t_f) \times L^\infty(t_0, t_f)$). Let P be a positive integer and denote $(t_f - t_0)/P$ by τ_1. In order to time-discretize (17) and (18), we advocate the following implicit-explicit scheme:

$$w^0 = w_0, \tag{19}$$

for $p = 1, \ldots, P$, w^{p-1} being known, solve

$$m\frac{w^p - w^{p-1}}{\tau_1} + c\ \mathrm{sgn}(w^p) = c\gamma\left(w^{p-1}\right) + f^p, \tag{20}$$

where $f^p = f(t_0 + p\tau_1)$ (or an approximation of it). An equivalent formulation of (20) is given by

$$\begin{aligned} m\frac{w^p - w^{p-1}}{\tau_1} + c\lambda^p &= c\gamma\left(w^{p-1}\right) + f^p, \\ \lambda^p w^p = |w^p|, \qquad |\lambda^p| &\le 1. \end{aligned} \tag{21}$$

Function $\xi \to m\xi + c\tau_1\,\mathrm{sgn}(\xi)$ being strictly monotone with range $\mathbf{R}$, problems (20) and (21) have unique solutions, $\forall\, \tau_1\ (\le t_f - t_0)$; we have

$$\begin{aligned} &w^p = 0, \qquad\qquad\qquad \text{if } |b^p| \le c\tau_1, \\ &w^p = \frac{(b^p - c\tau_1\,\mathrm{sgn}(b^p))}{m}, \qquad \text{if } |b^p| \ge c\tau_1, \end{aligned} \tag{22}$$

with $b^p = mw^{p-1} + c\tau_1\gamma(w^{p-1}) + \tau_1 f^p$. Once w^p is known, we obtain λ^p from the first equation in (21). Indeed, scheme (19),(20) is *unconditionally stable* and using again *compactness arguments*, we can easily show that

$$\lim_{\tau_1 \to 0} \max_{1 \le p \le P} |w^p - w(t_0 + p\tau_1)| = 0. \tag{23}$$

Remark 3. Define λ_{τ_1} by $\lambda_{\tau_1} = \sum_{p=1}^{P} \lambda^p \chi_p$, where χ_p is the characteristic function of $(t_0, t_f) \cap (t_0 + \tau_1(p - 1/2), t_0 + \tau(p + 1/2))$. We have $\lim_{\tau_1 \to 0} \lambda_{\tau_1} = \lambda$ in $L^\infty(t_0, t_f)$ weak-*.

4. NUMERICAL SOLUTION OF THE SUBPROBLEMS OF TYPE (12)

Problem (12) can be solved exactly; however, as a preparation to nonlinear and/or multidimensional variants, we shall briefly discuss its solution via (classical) difference schemes. Let Q be a positive integer and denote $\Delta t/Q$ by τ_2. With obvious notation (and $0 \le \alpha \le 1/2$), we approximate problem (12) by

$$x^{n+1,0} = x^{n+1/2}, \qquad x^{n+1,1} - x^{n+1,-1} = 2\tau_2 v^{n+1/2}, \tag{24}$$

for $q = 0, \ldots, Q+1$, $x^{n+1,q}$ and $x^{n+1,q-1}$ being known, solve

$$m\frac{x^{n+1,q+1} + x^{n+1,q-1} - 2x^{n+1,q}}{\tau_2^2} + k\left(\alpha x^{n+1,q+1} + (1-2\alpha)x^{n+1,q} + \alpha x^{n+1,q-1}\right) = 0, \quad (25)$$

$$x^{n+1} = x^{n+1,Q}, \qquad v^{n+1} = \frac{(x^{n+1,Q+1} - x^{n+1,Q-1})}{2\tau_2}. \quad (26)$$

It is well known (see, e.g., [1,2] and the references therein) that scheme (24),(25) is *unconditionally stable* if $1/4 \leq \alpha \leq 1/2$; if $0 \leq \alpha < 1/4$, "we" have stability provided Δt verifies $\Delta t < 1/\sqrt{(1/4-\alpha)k/m}$ (i.e., $\Delta t < 2\sqrt{m/k}$ if $\alpha = 0$).

5. NUMERICAL EXPERIMENTS

In order to validate the methodology discussed in the above sections, we consider two test problems with closed form solutions. The first test problem is a particular case of (17), i.e., a *pure friction problem*, while the second one is a particular case of (1), with $k > 0$. For both problems, operator γ is defined by (3).

5.1. First Test Problem

In (17), we take $t_0 = 0$, $t_f = 2$, $m = 1$, $c = 0.5$, $w_0 = 0$, and γ defined by (3) with $\beta = 1/3$ and $\epsilon = 1/10$; the forcing term is given by

$$f(t) = \begin{cases} 2\pi m \cos 2\pi t + c[1 - \gamma(\sin 2\pi t)], & \text{if } t \in \left(0, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right), \\ 0, & \text{if } t \in \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, 2\right). \end{cases}$$

With such f and w_0, the unique solution of problem (17) is given by $w(t) = (\sin 2\pi t)^+$ $(= \max(0, \sin 2\pi t))$, $\forall\, t \in [0,2]$. On Figure 1, we have shown the graph of the approximate solution computed with $\Delta t = 10^{-3}$. On Figure 2, we have represented on a log-scale the variation of the L^2-approximation error as a function of Δt. This figure clearly "suggests" *first-order accuracy*, for this test problem at least.

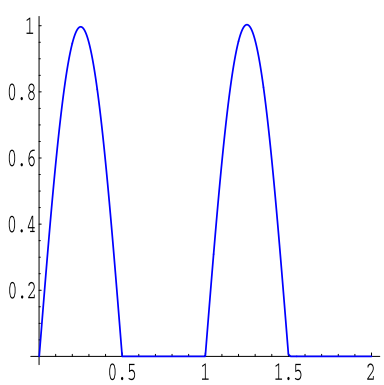

Figure 1. Test problem 1: graph of the computed solution.

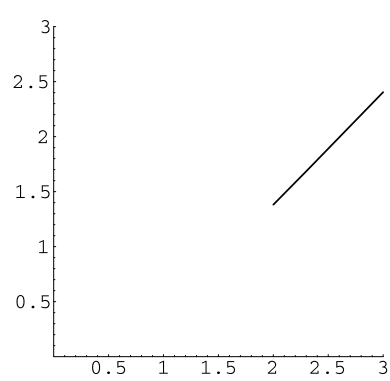

Figure 2. Test problem 1: variation of the L^2-error versus Δt (log-scale).

5.2. Second Test Problem

In (1), we take $T = 3$, $m = 1$, $c = 0.2$, $k = 1$, $x_0 = 0$, $v_0 = 0$, and γ is as in Section 5.1; this time, the forcing term is given by

$$f(t) = \begin{cases} 8m\pi^2 \cos 4\pi t + k \sin^2 2\pi t + c[1 - \gamma(2\pi \sin 4\pi t)], & \text{if } t \in \left(1, \frac{5}{4}\right) \cup \left(2, \frac{9}{4}\right), \\ 8m\pi^2 \cos 4\pi t + k \sin^2 2\pi t - c[1 + \gamma(2\pi \sin 4\pi t)], & \text{if } t \in \left(\frac{5}{4}, \frac{3}{2}\right) \cup \left(\frac{9}{4}, \frac{5}{2}\right), \\ -\frac{c}{2}, & \text{if } t \in \left(\frac{3}{2}, 2\right) \cup \left(\frac{5}{2}, 3\right). \end{cases}$$

For the above x_0, v_0, and f, the solution of problem (1) is given by $x(t) = (\sin 2\pi t)^{+2}$, $\forall\, t \in [1, 3]$. To solve problem (1), we have used the splitting scheme (9)–(12), the subproblems (10) and (11) being solved via schemes (19),(20) and (24)–(26), respectively. The following results have been obtained with $\tau_1 = \Delta t/10$ and $\tau_2 = \Delta t/2$. On Figures 3 and 4, we have shown the graphs of the approximation of x and $\dot{x}$, respectively, both obtained with $\Delta t = 10^{-3}$. Finally, on Figures 5 and 6, we have visualized again on a log-scale, the variations of the L^2-errors for x and $\dot{x}$ versus Δt. Once again, we observe *first-order accuracy*.

Remark 4. Using the *symmetrized scheme* (13)–(16) does not improve accuracy; this is not surprising after all, since we are dealing with a *nonsmooth* model.

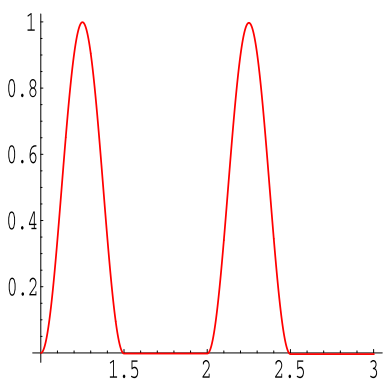

Figure 3. Test problem 2: graph of the computed x.

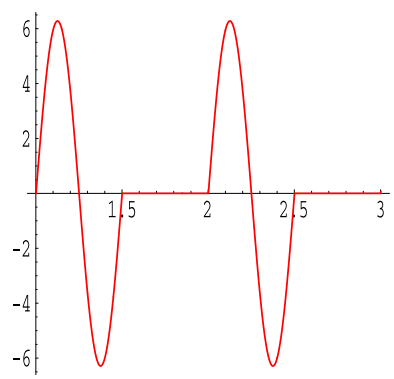

Figure 4. Test problem 2: graph of the computed $v = \dot{x}$.

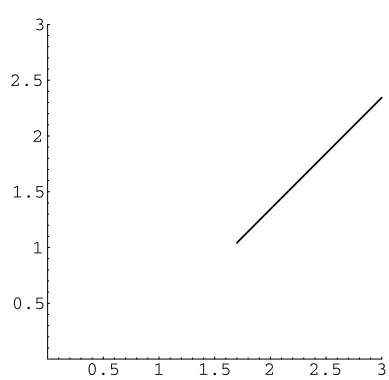

Figure 5. Test problem 2: L^2-error on x: variation versus Δt.

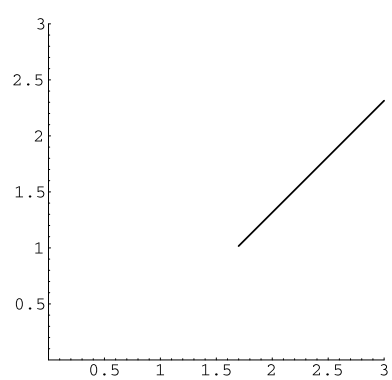

Figure 6. Test problem 2: L^2-error on $v = \dot{x}$: variation versus Δt.

6. FURTHER REMARKS

REMARK 5. From equation (4), it is clear that the accurate evaluation of the *friction force* requires the accurate evaluation of λ. This issue will be addressed in a forthcoming article together with the generalization of the techniques discussed here to multidimensional systems.

REMARK 6. The computational methods discussed in Section 3 can be easily generalized to the solution of the so-called (by NASA engineers) "gear box efficiency problem", a variant of problem (17),(18) defined as follows:

$$\begin{aligned} &m\dot{w} + c(\mathrm{sgn}(w) - \gamma(w)) + k(\delta)g(\delta w) = f, \qquad \text{on } (t_0, t_f), \\ &w(t_0) = w_0, \end{aligned} \tag{27}$$

where in (27):

(i) parameter δ is given in $\mathbf{R}$;
(ii) $k(\cdot)$ is an increasing odd function of δ vanishing at 0 and Lipschitz continuous over $\mathbf{R}$;
(iii) function g is of the following form:

$$g(\xi) = \frac{a+b}{2} + \frac{b-a}{2}\left[\mathrm{sgn}(\xi) - \gamma_{gb}(\xi)\right],$$

with $0 < a < b$ and function γ_{gb} of the same type than γ (see Section 2 for details).

The *monotonicity*, $\forall\, \delta \in \mathbf{R}$, of operator $w \to k(\delta)\,\mathrm{sgn}(\delta w)$, is the property making the above generalization possible.

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