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Numerical methods for a class of nonlinear integro-differential equations

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Abstract In a previous article (Glowinski, J. Math. Anal. Appl. 41, 67–96, 1973) the first author discussed several methods for the numerical solution of nonlinear equations of the integro-differential type with periodic boundary conditions. In this article we discuss an alternative methodology largely based on the Strang's symmetrized operator-splitting scheme. Several numerical experiments suggest that the new method is robust and accurate. It is also easier to implement than the various methods discussed by Glowinski in J. Math. Anal. Appl. 41, 67–96 (1973).

Keywords Integro-differential equations · Finite differences · Symmetrized operator-splitting schemes

Mathematics Subject Classification 65M99 · 65M20 · 65L06

1 Introduction

Motivated by the numerical solution of nonlinear integro-differential equations modeling nonlinear distortion correctors in high-power color TV transmitters, the first author discussed in [8] the finite difference approximation, and the iterative solution of nonlinear integro-differential equations of the following type:

$$\frac{du}{dx} + \phi(u) + Au = f \quad \text{in } (0,1), \qquad u(0) = u(1), \tag{1.1}$$

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where

(1) $\phi : (\alpha, \beta) \mapsto \mathbf{R}$ is continuous and non-decreasing (with $-\infty \le \alpha < \beta \le +\infty$ and $\phi(\alpha) = -\infty$, $\phi(\beta) = +\infty$).
(2) A is the *integral* operator defined by

$$(Av)(x) = \int_0^1 a(x, y)v(y)dy, \quad x \in (0, 1).$$

We assume that A is positive semi-definite, that is

$$\int_0^1 v(x)dx \int_0^1 a(x, y)v(y)dy \ge 0, \quad \forall v.$$

When, several decades ago, the first author had to address the numerical solution of problems such as (1.1), he considered applying the methods discussed in [1–3]. However, these methods had troubles handling those situations where $(\alpha, \beta) \ne \mathbf{R}$. It is also worth noticing that in order to solve an operator equation such as $A(u) = 0$, where A maps an Hilbert space H into itself, the author of [1] has been advocating (as many have done before and after) the following (Picard) algorithm,

$$u^0 \text{ given in } H. \tag{1.2}$$

For $n \ge 0$, u^n being known, we compute u^{n+1} via

$$u^{n+1} = u^n - \rho A\big(u^n\big), \tag{1.3}$$

with (typically) $\rho > 0$ and not too large. Actually, algorithm (1.2), (1.3) can be obtained by applying the forward Euler scheme to the time discretization of the following initial value problem in H,

$$\begin{cases} \frac{du}{dt} + A(u) = 0 \quad \text{in } (0, +\infty), \\ u(0) = u^0. \end{cases} \tag{1.4}$$

The method discussed in this article is in the same spirit, but, due to the fact that $(\alpha, \beta) \ne \mathbf{R}$, we have used a more implicit variant of (1.2) and (1.3), taking advantage, via operator-splitting, of the decomposition properties of the operator on the left-hand side of (1.1).

Indeed, in order to overcome the difficulties associated with ϕ when $(\alpha, \beta) \ne \mathbf{R}$, we advocated in [8], for the numerical solution of problem (1.1), a methodology combining a Galerkin approximation with finite difference methods, the resulting nonlinear system being solved by an *alternating direction algorithm* of the *Douglas-Rachford* type (see, e.g., [10], Chap. 2, for a discussion of *Douglas-Rachford* and related alternating direction algorithms, and further references). Actually, the numerical method discussed in [8] can be viewed as an approximate solution method for the following initial value problem (of the *hyperbolic* type):

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \phi(u) + Au = f & \text{in } (0, 1) \times (0, +\infty), \\ u(0, t) = u(1, t), & t \in (0, +\infty), \\ u(x, 0) = u_0(x), & x \in (0, 1). \end{cases} \tag{1.5}$$

The solution of problem (1.1) is a steady state solution of system (1.5) obtained via time integration from $t = 0$ to $t = +\infty$. Since the publication of [8], considerable progresses have been achieved concerning the use of operator-splitting methods for the solution of initial value problems involving several operators, such as in system (1.5). (Related publications are [9–16], among many others; see also the references therein.) The main goal of this article is to take advantage of these progresses to revisit the numerical solution of problem (1.1), applying to system (1.5) an operator-splitting scheme of the *Strang's symmetrized type*, optimally suited to the decomposition properties of the operator in problem (1.1) and system (1.5) which is clearly the sum of three operators.

In the following sections, we will discuss the application of the Strang's symmetrized operator-splitting scheme to the solution of (1.1), via (1.5), and present the numerical results obtained when applying the novel methodology to the test problems considered in [8].

In [8], one proved the existence and uniqueness of a solution to problem (1.1), assuming that:

(1) The kernel of the integral operator A, that is the function $a(\cdot,\cdot)$, belongs to $L^\infty((0,1)^2)$;
(2) $f \in L^\infty(0,1)$;
(3) Either the function ϕ is strictly increasing or the integral operator A is positive definite (of course, both properties can be verified simultaneously).

In this article, we are going to assume stronger properties for A and f, namely

$$f \in C^0[0,1], \qquad f(0) = f(1), \tag{1.6}$$

and the kernel of the integral operator A is such that

$$Av \in C^0[0,1], \qquad (Av)(0) = (Av)(1), \quad \forall v \in L^1(0,1). \tag{1.7}$$

One of the simplest kernels leading to a positive definite integral operator A verifying (1.7) is defined by

$$\begin{cases} a(x,y) = \frac{1}{e-1}e^{y-x+1}, & \text{if } 0 \le y < x \le 1, \\ a(x,y) = \frac{1}{e-1}e^{y-x}, & \text{if } 0 \le x < y \le 1. \end{cases} \tag{1.8}$$

If we consider the following two-point boundary value problem

$$\begin{cases} \frac{du}{dx} + u = f & \text{in } (0,1), \\ u(0) = u(1), \end{cases} \tag{1.9}$$

with $f \in L^1(0,1)$, then the unique continuous solution of Eq. (1.9) is given by

$$u(x) = \int_0^1 a(x,y) f(y)dy, \quad \forall x \in [0,1], \tag{1.10}$$

with $a(x,y)$ given by (1.8).

Remark 1.1 The test problems considered in Sect. 7 are essentially borrowed from [8]; they all verify the assumption (1.7). Concerning f, the test problems 1, 2 and 3 verify (1.6); the discontinuous function f associated with the fourth test problem can be easily approximated by continuous functions verifying (1.6).

Remark 1.2 The numerical solution of linear and nonlinear integral and integro-differential equations has motivated a large number of publications. In addition to [2, 3] and [1], let us mention among many others [4, 6, 7, 18, 19], and [5] (see also the references therein). However, to the best of our knowledge, the method discussed in this article seems to be ideally suited to the problem under consideration. In particular, it can handle easily those situations where $(\alpha, \beta) \neq \mathbf{R}$ and does not require the solution of large systems of linear and nonlinear equations.

2 On the Strang's symmetrized scheme

To the best of our knowledge, the *Strang's symmetrized scheme* was introduced in [17]. Applied to the solution of the *initial value problem*

$$\begin{cases} \frac{d\varphi}{dt} + B(\varphi, t) = 0 \quad \text{in } (0, T), \\ \varphi(0) = \varphi_0, \end{cases} \tag{2.1}$$

with $0 < T \leq +\infty$ and $B = B_1 + B_2$, the *Strang's symmetrized scheme* takes the following form (with $\Delta t (> 0)$ a time discretization step, and $t^{n+\alpha} = (n+\alpha)\Delta t$):

$$\varphi^0 = \varphi_0. \tag{2.2}$$

For $n \geq 0$, we obtain φ^{n+1} from φ^n via

$$\begin{cases} \varphi^{n+1/2} = \varphi(t^{n+1/2}), & \varphi \text{ being the solution of} \\ \frac{d\varphi}{dt} + B_1(\varphi, t) = 0 & \text{in } (t^n, t^{n+1/2}),\ \varphi(t^n) = \varphi^n, \end{cases} \tag{2.3}$$

$$\begin{cases} \hat{\varphi}^{n+1/2} = \varphi(\Delta t), & \varphi \text{ being the solution of} \\ \frac{d\varphi}{dt} + B_2(\varphi, t^{n+1/2}) = 0 & \text{in } (0, \Delta t),\ \varphi(0) = \varphi^{n+1/2}, \end{cases} \tag{2.4}$$

$$\begin{cases} \varphi^{n+1} = \varphi(t^{n+1}), & \varphi \text{ being the solution of} \\ \frac{d\varphi}{dt} + B_1(\varphi, t) = 0 & \text{in } (t^{n+1/2}, t^{n+1}),\ \varphi(t^{n+1/2}) = \hat{\varphi}^{n+1/2}. \end{cases} \tag{2.5}$$

Remark 2.1 As written, scheme (2.2)–(2.5) is not fully constructive, since we still have to solve the initial value problems in (2.3), (2.4) and (2.5). Several aspects of this most important issue will appear more clearly when applying scheme (2.2)–(2.5) to the solution of problem (1.1).

Remark 2.2 Assuming that more than two operators are involved, such as $B = B_1 + B_2 + B_3$, we return easily to the two-operator situation by observing that $B = (B_1 + B_2) + B_3$, or $B = B_1 + (B_2 + B_3)$. By application of the Strang's symmetrized scheme, the first (respectively, the second) decomposition leads to 7 (respectively 5) fractional steps. When applying scheme (2.2)–(2.5) to the solution of problem (1.1), we will use the second decomposition in order to minimize the number of fractional steps.

3 Application of scheme (2.2)–(2.5) to the solution of problems (1.1) and (1.5)

In order to apply the material of Sect. 2 to the solution of (1.1), we rewrite (1.5) in the following more abstract form

$$\begin{cases} \frac{du}{dt} + B_1 u + B_2(u) + B_3 u = 0 & \text{in } (0, +\infty), \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where in (3.1):

(1) The linear operator B_1 is associated with $\frac{\partial u}{\partial x}$ and the periodic boundary conditions.
(2) The nonlinear operator B_2 is defined by $B_2(v)(x) \equiv \phi(v(x)) - f(x)$.
(3) The linear operator B_3 is defined by $(B_3 v)(x) \equiv \int_0^1 a(x,y)v(y)dy$.

Applying the symmetrized scheme (2.2)–(2.5) to the time-discretization of problem (1.5), using the decomposition $B = B_1 + (B_2 + B_3)$, we obtain (denoting by $v(t)$ the function $x \mapsto v(x,t)$):

$$u(0) = u_0. \tag{3.2}$$

For $n \geq 0$, we obtain u^{n+1} from u^n via the solution of

$$\begin{cases} u^{n+1/5} = u(t^{n+1/2}), \quad u \text{ being the solution of the initial value problem} \\ \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 & \text{in } (0,1) \times (t^n, t^{n+1/2}), \\ u(0,t) = u(1,t) & \text{in } (t^n, t^{n+1/2}), \\ u(t^n) = u^n, \end{cases} \end{cases} \tag{3.3}$$

$$\begin{cases} u^{n+2/5} = u(\frac{\Delta t}{2}), \quad u \text{ being the solution of the initial value problem} \\ \begin{cases} \frac{\partial u}{\partial t} + \phi(u) = f & \text{in } (0,1) \times (0, \Delta t/2), \\ u(0) = u^{n+1/5}, \end{cases} \end{cases} \tag{3.4}$$

$$\begin{cases} u^{n+3/5} = u(\Delta t), \quad u \text{ being the solution of the initial value problem} \\ \begin{cases} \frac{\partial u}{\partial t} + \int_0^1 a(x,y)u(y,t)dy = 0 & \text{in } (0,1) \times (0, \Delta t), \\ u(0) = u^{n+2/5}, \end{cases} \end{cases} \tag{3.5}$$

$$\begin{cases} u^{n+4/5} = u(\Delta t), \quad u \text{ being the solution of the initial value problem} \\ \begin{cases} \frac{\partial u}{\partial t} + \phi(u) = f & \text{in } (0,1) \times (\Delta t/2, \Delta t), \\ u(0) = u^{n+3/5}, \end{cases} \end{cases} \tag{3.6}$$

$$\begin{cases} u^{n+1} = u(t^{n+1}), \quad u \text{ being the solution of the initial value problem} \\ \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 & \text{in } (0,1) \times (t^{n+1/2}, t^{n+1}), \\ u(0,t) = u(1,t) & \text{in } (t^{n+1/2}, t^{n+1}), \\ u(t^{n+1/2}) = u^{n+4/5}. \end{cases} \end{cases} \tag{3.7}$$

In Sects. 4, 5 and 6, we will discuss the solution of the three types of initial value problems one encounters, when applying scheme (3.2)–(3.7) to the solution of problem (1.5).

Remark 3.1 The integral operator component of problem (1.1) is taken into account via (3.5). The initial value problem (3.5) is nothing but an integro-differential equation associated with a bounded linear operator; making the solution of (3.5) quite easy, as shown in Sect. 6.

Remark 3.2 Suppose that u_0 in (1.5) and (3.2) is continuous over [0, 1] and periodic. If the properties (1.6) and (1.7) of f and A are verified, we can easily show that, $\forall n \geq 0$, $u^{n+1/5}$, $u^{n+2/5}$, $u^{n+3/5}$, $u^{n+4/5}$, and u^{n+1} share the properties of continuity and periodicity of u_0. From now on, we will assume the continuity and periodicity of u_0 on [0, 1].

4 On the solution of the sub-problems in (3.3) and (3.7)

4.1 Generalities

The sub-problems in (3.3) and (3.7) are both of the following type

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 & \text{in } (0,1) \times (t_0, t_1), \\ u(0,t) = u(1,t) & \text{in } (t_0, t_1), \\ u(t_0) = w, \end{cases} \tag{4.1}$$

where, in (4.1), the function $w \in C_p^0[0,1] = \{\varphi | \varphi \in C^0[0,1], \varphi(0) = \varphi(1)\}$ and $0 < t_1 - t_0 < 1$. Since the general solution of the equation $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ is of the form $u(x,t) = g(x-t)$, we clearly have

$$u(x,t_1) = \begin{cases} w(x-(t_1-t_0)), & \text{if } x-(t_1-t_0) \in [0,1], x \in [0,1], \\ w(x-(t_1-t_0)+1), & \text{if } x-(t_1-t_0) < 0, x \in [0,1]. \end{cases} \tag{4.2}$$

4.2 Application to the solution of the initial value problems in (3.3) and (3.7)

Applying the relations (4.2) to (3.3) and (3.7), we obtain

$$u^{n+1/5}(x) = \begin{cases} u^n(x - \frac{\Delta t}{2}), & \text{if } x - \frac{\Delta t}{2} \in [0,1], x \in [0,1], \\ u^n(x + 1 - \frac{\Delta t}{2}), & \text{if } x - \frac{\Delta t}{2} < 0, x \in [0,1], \end{cases} \tag{4.3}$$

and similarly

$$u^{n+1}(x) = \begin{cases} u^{n+4/5}(x - \frac{\Delta t}{2}), & \text{if } x - \frac{\Delta t}{2} \in [0,1], x \in [0,1], \\ u^{n+4/5}(x + 1 - \frac{\Delta t}{2}), & \text{if } x - \frac{\Delta t}{2} < 0, x \in [0,1]. \end{cases} \tag{4.4}$$

4.3 On the full discretization of the initial value problems in (3.3) and (3.7)

Let I be a positive integer ($\gg 1$, in practice). We introduce $\Delta x = \frac{1}{I}$, $x_i = i\Delta x$, $\forall i = 1, \ldots, I$, and take $\Delta t = \Delta x$. In the following, u_i^n will denote an approximation of $u(i\Delta x, n\Delta t)$, with $u_0^n = u_I^n$ to respect the x-periodicity.

Concerning the computation of $u^{n+1/5}$ and u^{n+1}, from (3.3) and (3.7), we take advantage of (4.3) and (4.4) as follows:

Suppose that for $n \geq 0$, u^n(respectively, $u^{n+4/5}$) is approximated by $\{u_i^n\}_{i=0}^I$ (respectively, $\{u_i^{n+4/5}\}_{i=0}^I$) with $u_0^n = u_I^n$(respectively, $u_0^{n+4/5} = u_I^{n+4/5}$). We then have $u^{n+1/5} = u^{n+4/5}$ approximated by $\{u_i^{n+1/5}\}_{i=0}^I$ and $\{u_i^{n+1}\}_{i=0}^I$) given by

$$u_0^{n+1/5} = \frac{1}{2}\left(u_{I-1}^n + u_0^n\right), \qquad u_i^{n+1/5} = \frac{1}{2}\left(u_{i-1}^n + u_i^n\right), \quad \forall i = 1, \ldots, I \tag{4.5}$$

and

$$u_0^{n+1} = \frac{1}{2}\left(u_{I-1}^{n+4/5} + u_0^{n+4/5}\right), \qquad u_i^{n+1} = \frac{1}{2}\left(u_{i-1}^{n+4/5} + u_i^{n+4/5}\right), \quad \forall i = 1, \ldots, I \tag{4.6}$$

respectively.

Remark 4.1 We can divide (approximately) by 2 the computational time dedicated to the solution of the hyperbolic problems in (3.3) and (3.7). Indeed, it suffices to observe that (as an obvious consequence of (3.3)–(3.7) and (4.3), (4.4)), for $n \geq 1$, we can merge (4.6) (with n replaced by $n-1$) and (4.5), and replace them by

$$u_0^{n+1/5} = u_{I-1}^{(n-1)+4/5}, \qquad u_i^{n+1/5} = u_{i-1}^{(n-1)+4/5}, \quad \forall i = 1, \ldots, I. \tag{4.7}$$

Numerical results obtained using (4.7), instead of (4.5) and (4.6), will be presented in Sect. 7.6.

5 On the solution of the sub-problems in (3.4) and (3.6)

We keep the notation in Sect. 4.3. To compute the approximation $\{u_i^{n+2/5}\}_{i=0}^I$ (respectively, $\{u_i^{n+4/5}\}_{i=0}^I$) of $u^{n+2/5}$ (respectively, $u^{n+4/5}$), we replace the differential equation in (3.4) (respectively (3.6)) by

$$\begin{aligned} &\frac{u_i^{n+2/5} - u_i^{n+1/5}}{\Delta t/2} + \phi\left(\frac{u_i^{n+2/5} + u_i^{n+1/5}}{2}\right) = f(i\Delta x), \quad \forall i = 0, 1, \ldots, I-1, \\ &\text{with } u_I^{n+2/5} = u_0^{n+2/5} \end{aligned} \tag{5.1}$$

(respectively,

$$\begin{aligned} &\frac{u_i^{n+4/5} - u_i^{n+3/5}}{\Delta t/2} + \phi\left(\frac{u_i^{n+4/5} + u_i^{n+3/5}}{2}\right) = f(i\Delta x), \quad \forall i = 0, 1, \ldots, I-1, \\ &\text{with } u_I^{n+4/5} = u_0^{n+4/5}). \end{aligned} \tag{5.2}$$

The one variable equations in (5.1) and (5.2) are obtained by applying *one step* of an *implicit second order Runge-Kutta scheme* to the equation $\frac{\partial u}{\partial t} + \phi(u) = f$. Now, in order to solve these nonlinear equations, we observe that they are all of the following type,

$$\frac{X - Y}{\Delta t/2} + \phi\left(\frac{X + Y}{2}\right) = F, \tag{5.3}$$

with Y and F given in $\mathbf{R}$. It is then convenient to introduce the new unknown $Z = \frac{X+Y}{2}$, that is $X = 2Z - Y$. Equation (5.3) becomes

$$Z + \frac{\Delta t}{4}\phi(Z) = Y + \frac{\Delta t}{4}F. \tag{5.4}$$

From the properties of ϕ, the one variable equation (5.4) has a unique solution in (α, β). The numerical solution of one variable equations such as (5.4) has motivated a most abundant literature. Actually, the properties of ϕ strongly influence the methodology appropriate for the solution of (5.4), as we will see in Sect. 7 for the particular case where $\phi(Z) = \ln(1 + Z)$.

6 On the solution of the sub-problem in (3.5)

6.1 Generalities. Time-discretization

The initial value problems encountered in (3.5) are of the following type:

$$\begin{cases} \frac{\partial u}{\partial t} + \int_0^1 a(x, y)u(y, t)dy = 0 \quad \text{in } (0, 1) \times (t_0, t_1), \\ u(t_0) = w, \end{cases} \tag{6.1}$$

with $w \in C_p^0[0, 1]$. In order to time-discretize the initial value problem (6.1), we first introduce a positive integer $M(\geq 1)$ and $\tau = \frac{t_1 - t_0}{M}$. Next, we denote $t_0 + m\tau$ by t^m (that is, $t^M = t_1$) and time-discretize (6.1) using the following *second order explicit Runge-Kutta scheme*:

$$u^0 = w; \tag{6.2}$$

then, for $m = 1, \dots, M$, we compute u^m from u^{m-1} via

$$u^{m-1/2} = u^{m-1} - \frac{\tau}{2}\int_0^1 a(x, y)u^{m-1}(y)dy, \tag{6.3}$$

$$u^m = u^{m-1} - \tau\int_0^1 a(x, y)u^{m-1/2}(y)dy. \tag{6.4}$$

When applying the above Runge-Kutta scheme to the solution of the sub-problem in (3.5), we took $M = 1$. Relations (6.2)–(6.4) thus lead to

$$\hat{u}^{n+3/5} = u^{n+2/5} - \frac{\Delta t}{2}\int_0^1 a(x, y)u^{n+2/5}(y)dy, \tag{6.5}$$

$$u^{n+3/5} = u^{n+2/5} - \Delta t\int_0^1 a(x, y)\hat{u}^{n+3/5}(y)dy. \tag{6.6}$$

6.2 Full discretization

We assume that $a(\cdot,\cdot)$ is continuous over $[0,1]^2\backslash\{\{x,y\}|x=y, 0\le x\le 1\}$, and consider $v\in C_p^0[0,1]$ and w defined by $w(x)=\int_0^1 a(x,y)v(y)dy, \forall x\in[0,1]$. From the property (1.7) of the kernel $a(\cdot,\cdot)$, we have $w\in C_p^0[0,1]$. For $i=0,1,\dots,I$, we approximate $w(x_i)$ (with $x_i=i\Delta x$) by w_i defined (from the *trapezoidal rule*) by

$$\begin{cases} w_0=\frac{1}{2}\Delta x\sum_{j=0}^{I}\alpha_j(a_{0j}+a_{Ij})v(x_j), \\ w_j=\Delta x\sum_{j=0}^{I}\alpha_j a_{ij}v(x_j), \quad \text{if } 1\le i\le I-1, \\ w_I=w_0 \end{cases} \tag{6.7}$$

in order to force the (discrete) *periodicity*. In (6.7), we have

(1) $\alpha_0=\alpha_I=\frac{1}{2}, \alpha_i=1, \forall i=1,\dots,I-1$.

(2) $a_{ij}=\begin{cases} a(x_i,x_j), & \text{if } i\ne j, \\ \frac{1}{2}[a_+(x_i,x_i)+a_-(x_i,x_i)], & \text{if } i=j, \end{cases}$

where

$a_+(x_i,x_i)=\lim_{\substack{\xi\mapsto x_i\\ \xi>x_i}} a(\xi,\xi)$ and $a_-(x_i,x_i)=\lim_{\substack{\xi\mapsto x_i\\ \xi<x_i}} a(\xi,\xi)$.

Applying relations (6.7) to the full approximation of (6.5) and (6.6), we obtain, with obvious notation,

$$\begin{cases} \hat{u}_0^{n+3/5}=\frac{1}{2}(u_0^{n+2/5}+u_I^{n+2/5})-\frac{1}{4}\Delta t\Delta x\sum_{j=0}^{I}\alpha_j(a_{0j}+a_{Ij})u_j^{n+2/5}, \\ \hat{u}_i^{n+3/5}=u_i^{n+2/5}-\frac{1}{2}\Delta t\Delta x\sum_{j=0}^{I}\alpha_j a_{ij}u_j^{n+2/5}, \quad \text{if } 1\le i\le I-1, \\ \hat{u}_I^{n+3/5}=\hat{u}_0^{n+3/5}, \end{cases} \tag{6.8}$$

and

$$\begin{cases} u_0^{n+3/5}=\frac{1}{2}(u_0^{n+2/5}+u_I^{n+2/5})-\frac{1}{2}\Delta t\Delta x\sum_{j=0}^{I}\alpha_j(a_{0j}+a_{Ij})\hat{u}_j^{n+3/5}, \\ u_i^{n+3/5}=u_i^{n+2/5}-\Delta t\Delta x\sum_{j=0}^{I}\alpha_j a_{ij}\hat{u}_j^{n+3/5}, \quad \text{if } 1\le i\le I-1, \\ u_I^{n+3/5}=u_0^{n+3/5}, \end{cases} \tag{6.9}$$

respectively.

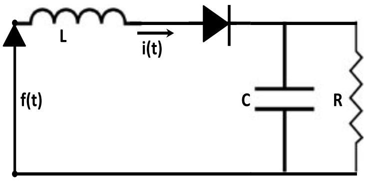
Relations (6.8) and (6.9) complete the description of the operator-splitting method, of the Strang symmetrized type, we advocate for the solution of (1.1), via the solution of the initial value problem (1.5). It will be validated in Sect. 7, by the solution of various test problems borrowed from [8].

7 Numerical experiments

7.1 Generalities

Three of the four test problems considered in this section are borrowed from [8]; these four problems are related to the *electrical circuit* shown in Fig. 1. The above circuit is excited by the periodic function of time f (a voltage here), its nonlinear

Fig. 1 A periodically excited nonlinear electrical circuit



behavior originating from insertion of the diode. In a well-chosen system of physical units, the periodic regime of the circuit is described by the following mathematical model verified by the electric current i:

$$\begin{cases} L\frac{di}{dt} + \ln(1+i) + Ki = f \quad \text{in } (0,1), \\ i(0) = i(1), \end{cases} \tag{7.1}$$

where, in (7.1), the linear operator K is defined by

$$(Kv)(t) = \int_0^1 k(t,\tau)v(\tau)d\tau, \quad \forall t \in [0,1],\ \forall v. \tag{7.2}$$

In (7.2), the kernel $k(\cdot,\cdot)$ is representative of the RC dipole and is given by

$$k(t,\tau) = \begin{cases} \alpha e^{-(t-\tau)/RC} & \text{if } t > \tau, \\ \alpha e^{-(1+t-\tau)/RC} & \text{if } t < \tau, \end{cases} \tag{7.3}$$

with $\alpha = \frac{1}{C(1-e^{-1/RC})}$; the integral operator K is positive definite. In the following we will replace t by x and τ by y, and use the notation of Sects. 1 to 6.

To visualize the convergence of the operator-splitting based time stepping method that we use to solve problem (1.1), we have introduced the two following residuals:

$$R_1^n = \Delta x \sum_{i=0}^{I-1} \left|u_i^{n+1} - u_i^n\right| \tag{7.4}$$

and

$$R_2^n = \sqrt{\Delta x \sum_{i=0}^{I-1} \left|u_i^{n+1} - u_i^n\right|^2}. \tag{7.5}$$

Remark 7.1 When applying the operator-splitting schemes (3.2)–(3.7) to the solution of the test problems considered below, we will have to solve (from (5.4), and with b given in $\mathbf{R}$) nonlinear problems of the following type:

$$Z + \frac{\Delta t}{4} \ln(1+Z) = b, \tag{7.6}$$

or equivalently

$$(1+Z)^{\frac{\Delta t}{4}} e^Z = e^b. \tag{7.7}$$

Equation (7.7) (and (7.6)) has a unique solution on the open half-line $(-1, +\infty)$; this solution verifies

$$\begin{cases} 0 < Z < b, & \text{if } b > 0, \\ Z = 0, & \text{if } b = 0, \\ -1 < Z < 0, & \text{if } b < 0. \end{cases} \tag{7.8}$$

From the relations (7.8), one can easily solve (7.7), using the *secant method* over either $[0, b]$ or $[-1, 0]$, depending of the sign of b. Actually, if $b > 0$, it may be more advantageous to solve (7.6), using the Newton's method initialized with b.

Remark 7.2 When solving (1.5) to capture asymptotically the solution of (1.1), we always initialized with $u_0 = 0$.

7.2 First test problem

The first test problem that we consider is defined by

$$\begin{cases} \frac{du}{dx} + \ln(1+u) + \alpha A u = f(x) & \text{in } (0, 1), \\ u(0) = u(1), \end{cases} \tag{7.9}$$

where in (7.9):

(1) The linear operator A is defined by $(Av)(x) = \int_0^1 a(x, y)v(y)dy, \forall x \in [0, 1], \forall v$, with $a(\cdot, \cdot)$ given by

$$a(x, y) = \begin{cases} e^{-(x-y)} & \text{if } x > y, \\ e^{-(1+x-y)} & \text{if } x < y. \end{cases} \tag{7.10}$$

(2) The forcing term f is given by

$$\begin{aligned} f(x) = &-1.96\pi \sin 2\pi x + \log(1 + 0.98 \cos 2\pi x) \\ &+ \frac{0.98\alpha}{1+4\pi^2}\left(1 - \frac{1}{e}\right)(\cos 2\pi x + 2\pi \sin 2\pi x). \end{aligned} \tag{7.11}$$

With the above data, the exact solution of problem (7.9) is given by

$$u(x) = 0.98 \cos 2\pi x. \tag{7.12}$$

When applying the computational methods discussed in Sects. 2 to 6 to the solution of problem (7.9), we took $\alpha = 5$ and $\Delta t = \Delta x$ varying from 5×10^{-2} to 2×10^{-3}.

On Fig. 2 we have visualized, on a log scale, the monotone decay of the residuals R_1^n and R_2^n for $\Delta t = \Delta x = 2.5 \times 10^{-3}$; we observe that these two functions of n are almost identical suggesting that the convergence is dominated by very few modes. For all the values of Δt that have been considered, we stopped iterating when $R_1^n \le 10^{-8}$. On Fig. 3 (left) we have compared the exact solution $u(x)$ with the computed solution $u^n(x)$, taking $\Delta t = \Delta x = 2.5 \times 10^{-3}$: the matching looks quite good, however, to make the comparison more quantitative, we have reported on the Fig. 3 (right), using a log-log scale, the variation versus Δx of the following approximation of the L^2-approximation error

$$ER_2 = \sqrt{\Delta x \sum_{i=0}^{I-1} \left| u(x_i) - u_i^{n_c} \right|^2}$$

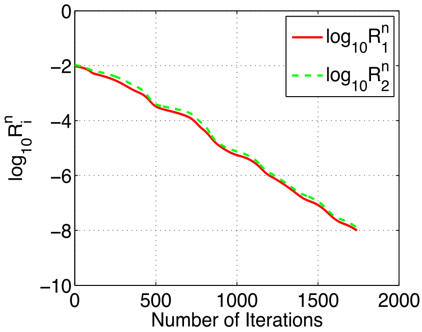


Fig. 2 First test problem: Variations of the residuals R_1^n and R_2^n versus the number of time steps ($\Delta t = \Delta x = 2.5 \times 10^{-3}$)

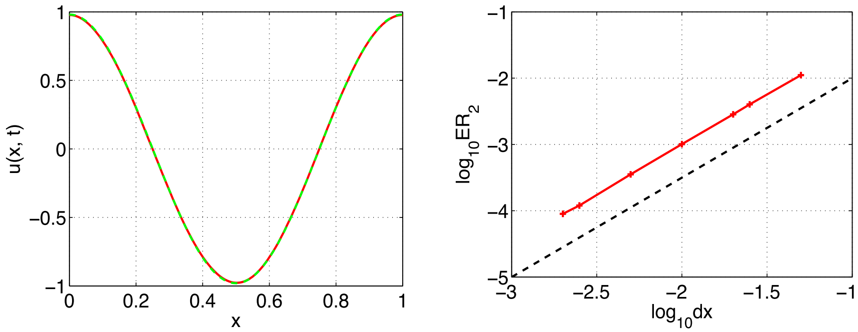


Fig. 3 First test problem: (*left*) comparison between the exact (–) and computed (- -) solutions ($\Delta t = \Delta x = 2.5 \times 10^{-3}$); (*right*) variation of the ER_2 versus Δx (–) (log-log scale; we used $\Delta t = \Delta x = 2 \times 10^{-3}, 2.5 \times 10^{-3}, 5 \times 10^{-3}, 10^{-2}, 2 \times 10^{-2}, 2.5 \times 10^{-2}, 5 \times 10^{-2}$)

where n_c is the number of iterations necessary to achieve convergence, according to the above stopping criterion; the above results strongly suggest that this error is $O(\Delta x^{3/2})$ (an evidence being the quasi-perfect parallelism between the graph of $\log_{10} ER_2$ and the (dashed) line of slope 3/2 going through the point at the intersection of the axes).

7.3 Second test problem

This test problem differs from the first one by the choice of f, everything else being the same including the parameter $\alpha(=5)$ and the stopping criterion. The function f is defined by

$$f(x) = 0.98\pi(4\cos 8\pi x - \sin 2\pi x) + \ln\big[1 + 0.49(\cos 2\pi x + \sin 8\pi x)\big] + \left(1 - \frac{1}{e}\right)\left[\frac{\alpha + 0.49}{1 + 4\pi^2}(\cos 2\pi x + 2\pi \sin 2\pi x) + \frac{\alpha + 0.49}{1 + 64\pi^2}(\sin 8\pi x - 8\pi \cos 8\pi x)\right]. \tag{7.13}$$

The corresponding exact solution is given by

$$u(x) = 0.49(\cos 2\pi x + \sin 8\pi x). \tag{7.14}$$

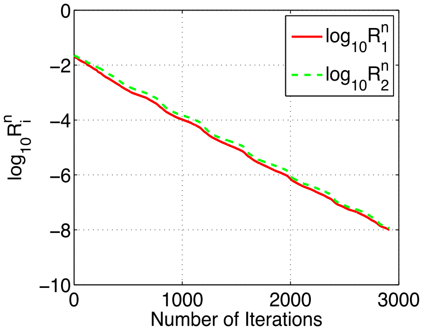


Fig. 4 Second test problem: Variations of the residuals R_1^n and R_2^n versus the number of time steps ($\Delta t = \Delta x = 2.5 \times 10^{-3}$)

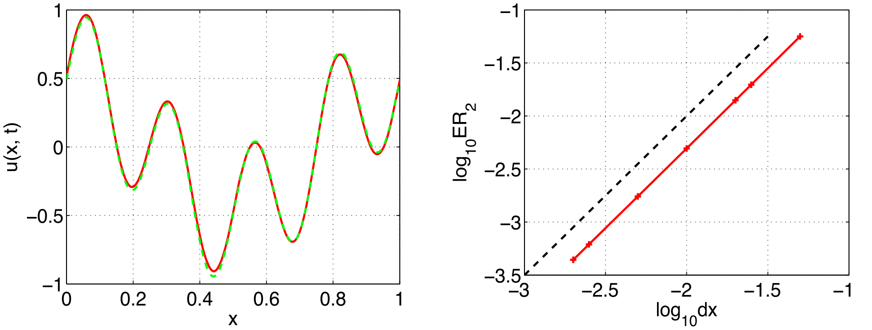


Fig. 5 Second test problem: (*left*) comparison between the exact (—) and computed (- -) solutions ($\Delta t = \Delta x = 2.5 \times 10^{-3}$); (*right*) the variation of ER_2 versus Δx (—) (log-log scale; We used $\Delta t = \Delta x = 2 \times 10^{-3}, 2.5 \times 10^{-3}, 5 \times 10^{-3}, 10^{-2}, 2 \times 10^{-2}, 2.5 \times 10^{-2}, 5 \times 10^{-2}$)

On Fig. 4 we have visualized the decay of the residuals R_1^n and R_2^n for $\Delta t = \Delta x = 2.5 \times 10^{-3}$. On Fig. 5 (left) we have compared the exact solution with the one computed with $\Delta t = \Delta x = 2.5 \times 10^{-3}$: again, the matching looks quite good. We have reported on the Fig. 5 (right) the variation of the L^2-approximation error versus Δx. The above results strongly suggest that, one more time, this error is $O(\Delta x^{3/2})$.

7.4 Third test problem

The third test problem is still described by (7.9), (7.10), with $\alpha = 1$ and 125, and f defined by

$$f(x) = 10 \cos 2\pi x. \tag{7.15}$$

For $\alpha = 1$ (resp., $\alpha = 125$), we have used $\Delta t = \Delta x = 10^{-2}$ (resp., 2.5×10^{-3}) and $R_1^n \leq 10^{-8}$ (resp., 10^{-6}), as stopping criterion. We have visualized on Fig. 6 the decay of the residuals R_1^n and R_2^n for $\alpha = 1$ (left) and $\alpha = 125$ (right); the slower convergence associated with $\alpha = 125$ originates from stronger nonlinear effects which reduce the smoothness of the solution and make less accurate the solution of the nonlinear problems (7.7) when u is close to -1. The loss of smoothness and the increased strength of the nonlinear effects, when one goes from $\alpha = 1$ to $\alpha = 125$, appear clearly on Fig. 7, where we have visualized the computed solutions.

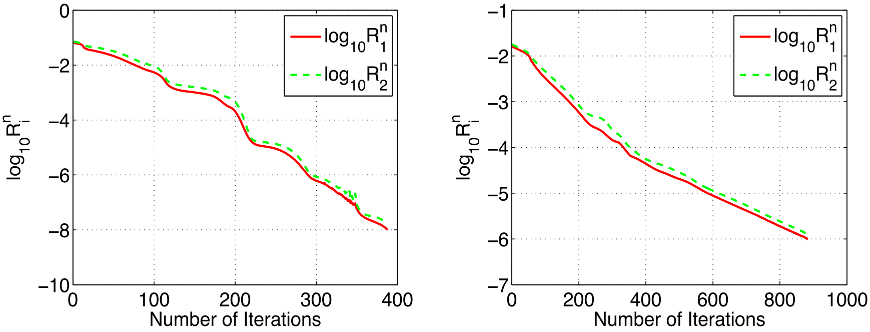


Fig. 6 Third test problem: Variations of the residuals R_1^n and R_2^n versus the number of time steps; (*left*) $\alpha = 1$, $\Delta t = \Delta x = 10^{-2}$, and (*right*) $\alpha = 125$, $\Delta t = \Delta x = 2.5 \times 10^{-3}$

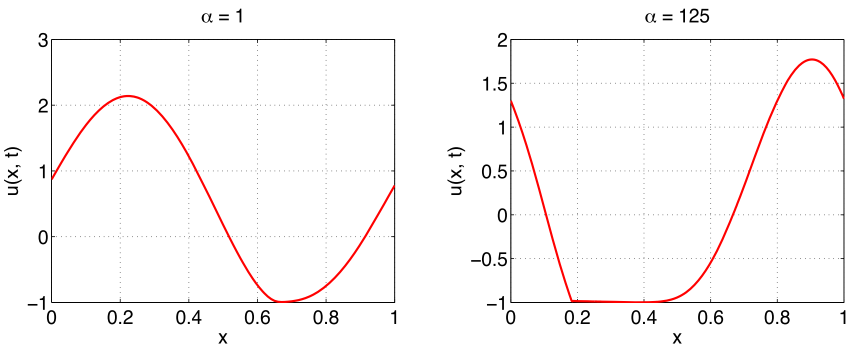


Fig. 7 Third test problem: Graphs of the computed solutions; (*left*) $\alpha = 1$, $\Delta t = \Delta x = 10^{-2}$, and (*right*) $\alpha = 125$, $\Delta t = \Delta x = 2.5 \times 10^{-3}$

7.5 The fourth test problem

The fourth test problem is still described by (7.9), (7.10) with $\alpha = 5$ and f is given by

$$f(x) = \begin{cases} \beta, & \text{if } 0 \leq x < \frac{1}{2}, \\ -\beta, & \text{if } \frac{1}{2} < x \leq 1. \end{cases} \tag{7.16}$$

The function f defined by (7.16) is neither continuous nor periodic. In order to apply to the numerical solution of this 4th test problem, the methods discussed in Sects. 2 to 6, for each value of Δx ($\ll 1$, in practice) we are going to approximate f by the continuous and periodic function $f_{\Delta x}$ defined by

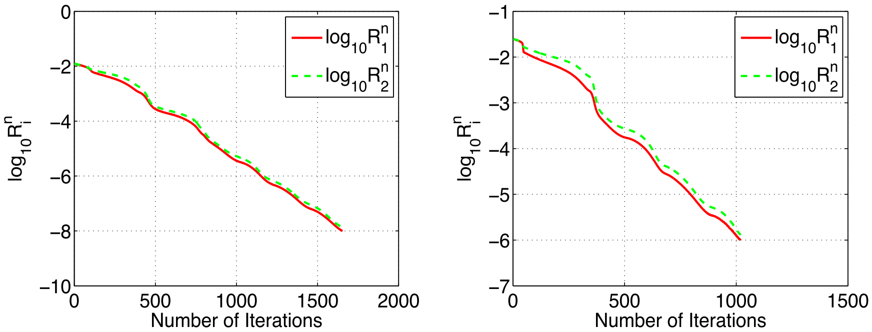


Fig. 8 Fourth test problem: Variations of the residuals R_1^n and R_2^n versus the number of time steps. (*left*) $\beta = 5$, $\Delta t = \Delta x = 2.5 \times 10^{-3}$; (*right*) $\beta = 10$, $\Delta t = \Delta x = 2.5 \times 10^{-3}$

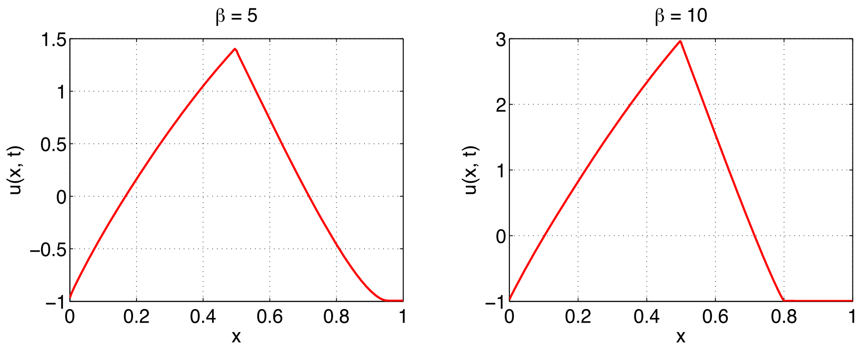


Fig. 9 Four test problem: Graphs of the computed solutions; (*left*) $\beta = 5$, $\Delta t = \Delta x = 2.5 \times 10^{-3}$; (*right*) $\beta = 10$, $\Delta t = \Delta x = 2.5 \times 10^{-3}$

$$f_{\Delta x}(x) = \begin{cases} \frac{\beta}{\Delta x} x, & \text{if } 0 \leq x \leq \Delta x, \\ \beta, & \text{if } \Delta x \leq x \leq \frac{1}{2} - \Delta x, \\ \frac{\beta}{\Delta x}(\frac{1}{2} - x), & \text{if } \frac{1}{2} - \Delta x \leq x \leq \frac{1}{2} + \Delta x, \\ -\beta, & \text{if } \frac{1}{2} + \Delta x \leq x \leq 1 - \Delta x, \\ \frac{\beta}{\Delta x}(x - 1), & \text{if } 1 - \Delta x \leq x \leq 1. \end{cases} \tag{7.17}$$

The numerical experiments associated with the 4th test problem have been carried out with $\beta = 5$ and 10. For $\Delta t = \Delta x = 2.5 \times 10^{-3}$, we have reported on Fig. 8, the variation versus n of the residuals R_1^n and R_2^n, taking as stopping criterion $R_1^n \leq 10^{-8}$ (resp., 10^{-6}) if $\beta = 5$ (resp., 10). As for the 3rd test problem, the stronger nonlinear effects associated with large values of β slow down the convergence. Another evidence of the stronger nonlinear effects associated with $\beta = 10$ appears clearly, when comparing the computed solutions reported in Fig. 9.

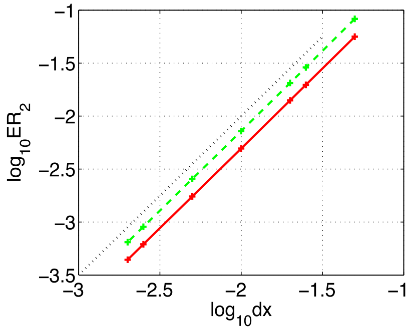


Fig. 10 Second test problem. Comparison of the discrete L^2-errors ER_2 between the methods with (−) and without (- -) averaging

7.6 Following on Remark 4.1

When inspecting the results obtained for the first and second test problems (see Sects. 7.2 and 7.3), which have known closed form solutions, we observe that the discrete analogue ER_2 of the $L^2(0,1)$-approximation error was behaving like $\Delta x^{3/2}$, modulo multiplication by a positive constant (see Figs. 3 and 5). The Strang's scheme being in principle second order accurate, we were wondering if this accuracy loss was coming of the averaging procedure taking place in (4.5) and (4.6). In order to check this assumption, we followed the suggestion of Remark 4.1 and replaced (4.5), (4.6) by (4.7) in the fully discrete Strang's scheme. When applied to the first and second test problems, the numerical results showed there was no improvement since the discrete L^2-approximation error was still no better than $\Delta x^{3/2}$, with in fact larger values. On Fig. 10, we have compared, for the second test problem, the variations of ER_2 versus Δx for the discrete solutions obtained with and without averaging. A possible explanation of the better performances of the method with averaging, is that the corresponding algorithm has stronger damping properties for the high modes of $u(n\Delta t) - u^n_{\Delta x}$ than the algorithm without averaging.

8 Further comments

The main goal of this article was to discuss an alternative to the methods discussed in [8], for the numerical solution of problem (1.1). The method discussed in Sects. 2 to 6 is robust, accurate and relatively simple to implement; nowhere it requires the solution of linear systems. We think that the centered scheme based method in [8]—second order accurate in principle—would have been competitive in terms of accuracy and stability; however, it would have been more complicated to implement since, at each (pseudo) time step, one has to solve a linear system associated with a *full* matrix (see [8] for details); let us mention that in [8] we took advantage of constant Δx and pseudo-time step to LU factorize once for all the above full matrix, this being facilitated by the fact that the integers I ($I = 1/\Delta x$) that we used were much smaller than the ones in the present article.

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