# OPERATOR SPLITTING METHOD FOR FRICTION CONSTRAINED DYNAMICAL SYSTEMS 

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#### Abstract

In a previous article [1] the time-discretization of those relations modeling a class of dynamical systems with friction was discussed. The main goal of this article is to address similar problems using a more sophisticated friction model giving a better description of the system behavior particularly when the velocities are close to zero. These investigations are motivated by the need for more accurate friction models in the software simulating the motion of mechanical systems, such as the remote manipulators of the Space Shuttle or of the International Space Station. In this article, we discuss the methods in the case of higher number of degrees of freedom elasto-dynamical systems, and the special case of one degree of freedom. The content can be summarized as follows: We discuss first models of the constrained motions under consideration, including a rigorous formulation involving a kind of dynamical multiplier. An iterative method allowing the computation of this multiplier will be discussed. Next, in order to treat friction, we introduce an implicit/explicit numerical scheme which is unconditionally stable, and easy to implement and generalize to more complicated systems. Indeed the above scheme can be coupled, via operator-splitting, to schemes classically used to solve differential equations from frictionless elasto-dynamics. The above schemes are validated through numerical experiments.


1. Introduction. Motivated by the real time simulation of elasto-dynamical systems with friction we introduced in [1], a family of numerical schemes take advantage of the existence of a friction multiplier. Discrepancies between simulations and real life results lead engineers to refine their friction models in order to improve simulation quality, particularly at relatively very low velocities, that is, when friction forces dominate the dynamics of the system under consideration.

[^0]2. Modeling of friction constrained motions: Splitting of the model. Some remote manipulator system simulators use finite number of degrees of freedom models, like the one below to describe friction constrained motions:
\[

\left\{$$
\begin{array}{l}
M \ddot{X}+A X+C(\operatorname{sgn}(\dot{X})-\gamma(\dot{X}))=f \quad \text { on }(0, T),  \tag{2.1}\\
X(0)=X_{0}, \dot{X}(0)=V_{0},
\end{array}
$$\right.
\]

where in equation (2.1): $X$ is a displacement (here $X(t) \in \mathbf{R}^{d}$ ), the mass matrix $M$ is symmetric and positive definite, the stiffness matrix $A$ is symmetric and positive semi-definite, the friction matrix $C$ is diagonal, i.e. $C=\operatorname{diag}\left(c_{1}, \cdots, c_{d}\right)$, with $c_{i} \geq 0, \forall i=1, \cdots, d$ and $\sum_{i=1}^{d} c_{i}>0, \operatorname{sgn}(V)=\left\{\operatorname{sgn}\left(v_{i}\right)\right\}_{i=1}^{d}, \forall V=\left\{v_{i}\right\}_{i=1}^{d} \in \mathbf{R}^{d}$, $\gamma(V)=\left\{\gamma_{i}\left(v_{i}\right)\right\}_{i=1}^{d}, \quad \forall V=\left\{v_{i}\right\}_{i=1}^{d} \in \mathbf{R}^{d}, \gamma_{i}$ being a nondecreasing Lipschitz continuous function vanishing at 0 and such that $\lim _{\xi \mapsto \pm \infty} \gamma_{i}(\xi)= \pm \beta_{i}$, with $0<$ $\beta_{i}<1$. Typical functions $\gamma_{i}$ are provided by

$$
\begin{equation*}
\gamma_{i}(\xi)=\beta_{i} \xi / \sqrt{\varepsilon^{2}+\xi^{2}} \tag{2.2}
\end{equation*}
$$

or

$$
\begin{equation*}
\gamma_{i}(\xi)=\beta_{i} \xi / \varepsilon \text { if }|\xi| \leq \varepsilon, \gamma_{i}(\xi)=\beta_{i} \operatorname{sgn}(\xi) \text { if }|\xi| \geq \varepsilon \tag{2.3}
\end{equation*}
$$

Operator $\gamma$ has been introduced to take into consideration the following well known fact: When there is dry friction, the force necessary to put the system into motion, starting from rest, is higher than the one necessary to maintain the motion. $f$ is an external force, $X_{0}, V_{0} \in \mathbf{R}^{d}$. Following [3], A rigorous equivalent formulation of (2.1) is given by

$$
\left\{\begin{array}{lr}
\dot{X}=V & \text { on }(0, T)  \tag{2.4}\\
M \dot{V}+A X+C(\lambda-\gamma(V))=f & \text { on }(0, T) \\
C \lambda(t) \cdot V(t)=\sum_{i=1}^{d} c_{i}\left|v_{i}(t)\right|, \lambda(t) \in \Lambda & \text { a.e. on }(0, T), \\
X(0)=X_{0}, V(0)=V_{0} &
\end{array}\right.
$$

with $\Lambda$ the closed convex non-empty subset of $\mathbf{R}^{d}$ defined by

$$
\Lambda=\left\{\mu\left|\mu \in \mathbf{R}^{d},\left|\mu_{i}\right| \leq 1, \forall i=1, \cdots, d\right\}\right.
$$

and $a \cdot b=\sum_{i=1}^{d} a_{i} b_{i}, \forall a=\left\{a_{i}\right\}_{i=1}^{d}, b=\left\{b_{i}\right\}_{i=1}^{d} \in \mathbf{R}^{d}$. The vector-valued function $C(\lambda-\gamma(V))$ models the friction forces present in the system. Suppose that $T$ is finite and let $\Delta t=T / N$. In order to solve problem (2.4), we advocate the following Lie's scheme (where $t^{n}=n \tau$ ):

$$
\begin{equation*}
X^{0}=X_{0}, V^{0}=V_{0} \tag{2.5}
\end{equation*}
$$

for $n=1, \cdots, N, X^{n}$ and $V^{n}$ being known, solve

$$
\begin{cases}M \dot{V}+C(\lambda-\gamma(V))=f & \text { on }\left(t^{n}, t^{n+1}\right),  \tag{2.6}\\ C \lambda(t) \cdot V(t)=\sum_{i=1}^{d} c_{i}\left|v_{i}(t)\right|, & \lambda(t) \in \Lambda \text { a.e. on }\left(t^{n}, t^{n+1}\right), \\ \dot{X}=0 & \text { on }\left(t^{n}, t^{n+1}\right), \\ V\left(t^{n}\right)=V^{n}, X\left(t^{n}\right)=X^{n}, & \end{cases}
$$

and set

$$
\begin{equation*}
V^{n+1 / 2}=V\left(t^{n+1}\right), X^{n+1 / 2}=X^{n} \tag{2.7}
\end{equation*}
$$

next solve

$$
\begin{cases}M \dot{V}+A X=0 & \text { on }\left(t^{n}, t^{n+1}\right)  \tag{2.8}\\ \dot{X}=V & \text { on }\left(t^{n}, t^{n+1}\right) \\ V\left(t^{n}\right)=V^{n+1 / 2}, & X\left(t^{n}\right)=X^{n+1 / 2}\end{cases}
$$

and set

$$
\begin{equation*}
V^{n+1}=V\left(t^{n+1}\right), X^{n+1}=X\left(t^{n+1}\right) \tag{2.9}
\end{equation*}
$$

Problem (2.8), the elastic step, is equivalent to

$$
\left\{\begin{array}{l}
M \ddot{X}+A X=0 \text { on }\left(t^{n}, t^{n+1}\right),  \tag{2.10}\\
X\left(t^{n}\right)=X^{n+1 / 2}, \dot{X}\left(t^{n}\right)=V^{n+1 / 2}
\end{array}\right.
$$

while (2.9) reads as

$$
\begin{equation*}
X^{n+1}=X\left(t^{n+1}\right), V^{n+1}=\dot{X}\left(t^{n+1}\right) \tag{2.11}
\end{equation*}
$$

Problems (2.8), (2.10) is a standard one whose numerical solution is a well documented topic which is dicussed in section 5 . On the other hand, solving problem (2.6), the friction step, is a more critical issue which is the main study of this article and is addressed in the following section.
3. Time-discretization of problem (2.6). Problem (2.6) is a special case of

$$
\begin{cases}M \dot{W}+C(\lambda-\gamma(W))=f & \text { on }\left(t_{0}, t_{f}\right)  \tag{3.12}\\ C \lambda(t) \cdot W(t)=\sum_{i=1}^{d} c_{i}\left|w_{i}(t)\right|, & \lambda(t) \in \Lambda \text { a.e. on }\left(t_{0}, t_{f}\right) \\ W\left(t_{0}\right)=W_{0}\end{cases}
$$

In order to time-discretize (3.12), we advocate the following implicit-explicit scheme (with $\left.\tau_{f}=\left(t_{f}-t_{0}\right) / P\right)$ :

$$
\begin{equation*}
W^{0}=W_{0} \tag{3.13}
\end{equation*}
$$

for $p=1, \cdots, P, W^{p-1}$ being known solve the following system of equations

$$
\left\{\begin{array}{l}
M \frac{W^{p}-W^{p-1}}{\tau_{f}}+C \lambda^{p}=C \gamma\left(W^{p-1}\right)+f^{p}  \tag{3.14}\\
C \lambda^{p} \cdot W^{p}=\sum_{i=1}^{d} c_{i}\left|w_{i}^{p}\right|, \lambda^{p} \in \Lambda,
\end{array}\right.
$$

where $f^{p}=f\left(t_{0}+p \tau_{f}\right)$. Using compactness arguments we can show that

$$
\lim _{\tau_{f} \rightarrow 0} \max _{1 \leq p \leq P}\left\|W^{p}-W\left(t_{0}+p \tau_{f}\right)\right\|=0
$$

and weak-* convergence to $\lambda$ in $L^{\infty}\left(t_{0}, t_{f} ; \mathbf{R}^{d}\right)$, for the sequence $\left\{\left\{\lambda^{p}\right\}_{p=1}^{P}\right\}_{P}$, where $\{W, \lambda\}$ is the unique solution of system (3.12). The iterative solution of system such as (3.14) will be briefly discussed in the following section. In particular, if $d=1$ computing the closed form solution of problem (3.14) is easy as follows. An equivalent formulation of (3.14) in one degree of freedom is given by

$$
\left\{\begin{array}{l}
m \frac{w^{p}-w^{p-1}}{\tau_{f}}+c \lambda^{p}=c \gamma\left(w^{p-1}\right)+f^{p}  \tag{3.15}\\
\lambda^{p} w^{p}=\left|w^{p}\right|,\left|\lambda^{p}\right| \leq 1
\end{array}\right.
$$

Function $\xi \rightarrow m \xi+c \tau_{f} \operatorname{sgn}(\xi)$ being strictly monotone with range $\mathbf{R}$, problems (3.15) have unique solutions, $\forall \tau_{f}\left(\leq t_{f}-t_{0}\right)$; we have

$$
\left\{\begin{array}{l}
w^{p}=0 \text { if }\left|b^{p}\right| \leq c \tau_{f},  \tag{3.16}\\
w^{p}=\left(b^{p}-c \tau_{f} \operatorname{sgn}\left(b^{p}\right)\right) / m \text { if }\left|b^{p}\right| \geq c \tau_{f}
\end{array}\right.
$$

with $b^{p}=m w^{p-1}+c \tau_{f} \gamma\left(w^{p-1}\right)+\tau_{f} f^{p}$. Once $w^{p}$ is known, we obtain $\lambda^{p}$ from the first equation in (3.15). Indeed, scheme(3.15) is unconditionally stable and using again compactness arguments, we can easily show that

$$
\begin{equation*}
\lim _{\tau_{f} \rightarrow 0} \max _{1 \leq p \leq P}\left|w^{p}-w\left(t_{0}+p \tau_{f}\right)\right|=0 \tag{3.17}
\end{equation*}
$$

Remark 1. Define $\lambda_{\tau_{f}}$ by $\lambda_{\tau_{f}}=\sum_{p=1}^{P} \lambda^{p} \chi_{p}$ where $\chi_{p}$ is the characteristic function of $\left(t_{0}, t_{f}\right) \cap\left(t_{0}+\tau_{f}(p-1 / 2), t_{0}+\tau_{f}(p+1 / 2)\right)$. We have $\lim _{\tau_{f} \rightarrow 0} \lambda_{\tau_{f}}=\lambda$ in $L^{\infty}\left(t_{0}, t_{f}\right)$ weak-*.
4. Iterative Solution of System (3.14). On the other hand, if $d \geq 2$, in multiple degrees of freedom, then we must rely on iterative techniques. Let $b^{p}=M W^{p-1}+$ $\tau_{f} C \gamma\left(W^{p-1}\right)+\tau_{f} f^{p}$, and drop the superscript $p$ in problem (3.14). It takes then the following form:

$$
\left\{\begin{array}{l}
M W+\tau_{f} C \lambda=b,  \tag{4.18}\\
C \lambda \cdot W=\sum_{i=1}^{d} c_{i}\left|w_{i}\right|, \lambda \in \Lambda .
\end{array}\right.
$$

A solution technique is provided by the following algorithm

$$
\begin{equation*}
\lambda^{0} \text { given in } \Lambda ; \tag{4.19}
\end{equation*}
$$

for $k \geq 0, \lambda^{k}$ being known, solve

$$
\begin{equation*}
M W^{k}=b-\tau_{f} C \lambda^{k} \tag{4.20}
\end{equation*}
$$

and update $\lambda^{k}$ via

$$
\begin{equation*}
\lambda^{k+1}=P_{\Lambda}\left(\lambda^{k}+\rho C W^{k}\right) \tag{4.21}
\end{equation*}
$$

In (4.21), the projection operator $P_{\Lambda}: \mathbf{R}^{d} \rightarrow \Lambda$ is defined by

$$
\begin{equation*}
P_{\Lambda}(\mu)=\left\{\min \left(1, \max \left(-1, \mu_{i}\right)\right)\right\}_{i=1}^{d}, \forall \mu=\left\{\mu_{i}\right\}_{i=1}^{d} \in \mathbf{R}^{d} . \tag{4.22}
\end{equation*}
$$

The set $\Lambda$ being closed, convex (and non-empty), operator $P_{\Lambda}$ is a contraction. Concerning the convergence of algorithm (4.19)-(4.21), we have the following theorem.

Theorem 1. Suppose that

$$
\begin{equation*}
0<\rho<\frac{2}{\tau_{f} \beta_{d}} \tag{4.23}
\end{equation*}
$$

where $\beta_{d}$ is the largest eigenvalue of matrix $M^{-1} C^{2}$; we have then, $\forall \lambda^{0} \in \Lambda$,

$$
\begin{equation*}
\lim _{k \rightarrow+\infty}\left\{W^{k}, \lambda^{k}\right\}=\{W, \lambda\}, \tag{4.24}
\end{equation*}
$$

where $\{W, \lambda\}$ is the solution of system (4.18).
An estimate of the speed of convergence of (4.19)-(4.21) will be given in a forthcoming publication [6].
Proof: The second relation in (4.18) is equivalent to

$$
\begin{equation*}
\lambda=P_{\Lambda}(\lambda+\rho C W), \forall \rho>0 . \tag{4.25}
\end{equation*}
$$

Let us denote $W^{k}-W$ and $\lambda^{k}-\lambda$ by $\bar{W}^{k}$ and $\bar{\lambda}^{k}$, respectively. Operator $P_{\Lambda}$ being a contraction in $\mathbf{R}^{d}$, by subtracting the first relation in (4.18) from (4.20), and (4.25) from (4.21), we have

$$
\begin{equation*}
M \bar{W}^{k}=-\tau_{f} C \bar{\lambda}^{k} \tag{4.26}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\bar{\lambda}^{k+1}\right\|_{\mathbf{R}^{d}} \leq\left\|\bar{\lambda}^{k}+\rho C \bar{W}^{k}\right\|_{\mathbf{R}^{d}} \tag{4.27}
\end{equation*}
$$

Combining both relations yields

$$
\begin{equation*}
\left\|\bar{\lambda}^{k}\right\|_{\mathbf{R}^{d}}^{2}-\left\|\bar{\lambda}^{k+1}\right\|_{\mathbf{R}^{d}}^{2} \geq-2 \rho C \bar{W}^{k} \cdot \bar{\lambda}^{k}-\rho^{2}\left\|C \bar{W}^{k}\right\|_{\mathbf{R}^{d}}^{2}=\frac{2 \rho}{\tau_{f}} M \bar{W}^{k} \cdot \bar{W}^{k}-\rho^{2}\left\|C \bar{W}^{k}\right\|_{\mathbf{R}^{d}}^{2} . \tag{4.28}
\end{equation*}
$$

Since $\|C z\|_{\mathbf{R}^{d}}^{2} \leq \beta_{d} M z \cdot z, \forall z \in \mathbf{R}^{d}$, with $\beta_{d}$ being the largest eigenvalue of matrix $M^{-1} C^{2}$, it follows from (4.28) that

$$
\begin{equation*}
\left\|\bar{\lambda}^{k}\right\|_{\mathbf{R}^{d}}^{2}-\left\|\bar{\lambda}^{k+1}\right\|_{\mathbf{R}^{d}}^{2} \geq \rho\left(\frac{2}{\tau_{f}}-\rho \beta_{d}\right) M \bar{W}^{k} \cdot \bar{W}^{k} \tag{4.29}
\end{equation*}
$$

The positive definiteness of $M$ and (4.29) imply that if (4.23) holds, then the sequence $\left\{\left\|\lambda^{k}\right\|_{\mathbf{R}^{d}}^{2}\right\}_{k}$ is decreasing, and therefore converging. Since bounded from below by zero. The convergence of the above sequence implies that the right-hand side in (4.29) converges to zero, implying in turn that $\bar{W}^{k} \mapsto 0$ (i.e. $W^{k} \mapsto W$ ). The convergence of $\left\{\lambda^{k}\right\}_{k}$ to $\lambda$ follows from that of $\left\{W^{k}\right\}_{k}$ and of (4.26).
5. Numerical solution of type (2.8) (that is (2.10)) subproblems. Problems (2.8),(2.10) are very classical ones. We shall briefly discuss their solution by (well known) finite difference schemes. Let $Q$ be a positive integer and $\tau_{2}=\Delta t / Q$. With obvious notation (and $0 \leq \alpha \leq 1 / 2$ ), we approximate problem (2.10) by

$$
\begin{equation*}
X^{n+1,0}=X^{n+1 / 2}, X^{n+1,1}-X^{n+1,-1}=2 \tau_{2} V^{n+1 / 2} \tag{5.1}
\end{equation*}
$$

for $q=0, \cdots, Q, X^{n+1, q}$ and $X^{n+1, q-1}$ being known, solve:

$$
\begin{gather*}
M \frac{X^{n+1, q+1}+X^{n+1, q-1}-2 X^{n+1, q}}{\tau_{2}^{2}} \\
+\quad A\left(\alpha X^{n+1, q+1}+(1-2 \alpha) X^{n+1, q}+\alpha X^{n+1, q-1}\right)=0  \tag{5.2}\\
X^{n+1}=X^{n+1, Q}, V^{n+1}=\left(X^{n+1, Q+1}-X^{n+1, Q-1}\right) / 2 \tau_{2} \tag{5.3}
\end{gather*}
$$

The schemes, (5.1)-(5.3), are used in both one degree and multiple degrees of freedom. It is well known (see, e.g., [1], [2] and the references therein) that scheme (5.1), (5.2) is unconditionally stable if $1 / 4 \leq \alpha \leq 1 / 2$; if $0 \leq \alpha<1 / 4$, one has stability provided that $\tau_{2}<1 / \sqrt{(1 / 4-\alpha) \nu_{d}}, \nu_{d}$ being the largest eigenvalue of matrix $M^{-1} A$ (i.e., $\tau_{2}<2 / \sqrt{\nu_{d}}$ if $\alpha=0$ ).
6. Numerical experiments. In order to validate the methodology discussed in the above sections, we consider three test problems with degrees of freedom from one to three. And, the first two test problems are given with closed form solutions.
6.1. Test problem with one degree of freedom: In (2.1), (due to one degree of freedom problem, the notations in (2.1) are lower-cased.) we take $T=3, m=$ $1, a=1, c=0.2, x_{0}=0, v_{0}=0$ and $\gamma$ is as in Section 2; the forcing term is given by
$f(t)= \begin{cases}8 m \pi^{2} \cos 4 \pi t+k \sin ^{2} 2 \pi t+c[1-\gamma(2 \pi \sin 4 \pi t)], & \text { if } t \in\left(1, \frac{5}{4}\right) \cup\left(2, \frac{9}{4}\right), \\ 8 m \pi^{2} \cos 4 \pi t+k \sin ^{2} 2 \pi t-c[1+\gamma(2 \pi \sin 4 \pi t)], & \text { if } t \in\left(\frac{5}{4}, \frac{3}{2}\right) \cup\left(\frac{9}{4}, \frac{5}{2}\right), \\ -c / 2 & \text { if } t \in\left(\frac{3}{2}, 2\right) \cup\left(\frac{5}{2}, 3\right) .\end{cases}$
For the above $x_{0}, v_{0}$ and $f$, the solution of problem (2.1) is given by $x(t)=$ $(\sin 2 \pi t)^{+2}, \forall t \in[1,3]$. To solve problem (2.1), we have used the splitting scheme (2.5) - (2.8), the subproblems (2.6) and (2.7) being solved via schemes (3.15) and (5.1) - (5.3), respectively. The following results have been obtained with $\tau_{1}=$ $\Delta t / 10$ and $\tau_{2}=\Delta t / 2$. On Figs. 1 and 2 , we have shown the graphs of the approximation of $x$ and $\dot{x}$, respectively, both obtained with $\Delta t=10^{-3}$. Finally, on Figs. 3 and 4, we have visualized again on a log-scale, the variations of the $L^{2}$-errors for


Figure 1. Test Problem 1:
Graph of the computed $x$.


Figure 2. Test Problem 1:
Graph of the computed $v=\dot{x}$.


Figure 3. Test Problem 1:


Figure 4. Test Problem 1: $L^{2}$-error on $x$ : variation versus $\Delta t L^{2}$-error on $v=\dot{x}$ : variation versus $\Delta t$
$x$ and $\dot{x}$ versus $\Delta t$. This figure clearly "suggests" first order accuracy, for this test problem at least.

Remark 2. From equation (2.4), it is clear that the accurate evaluation of the friction force requires the accurate evaluation of $\lambda$. This issue will be addressed in [6] as well.

Remark 3. The computational methods discussed in Section 3 can be easily generalized to the solution of the so-called (by NASA engineers) "gear box efficiency problem", a variant of problem (3.12) defined as follows:

$$
\left\{\begin{array}{l}
m \dot{w}+c(\operatorname{sgn}(w)-\gamma(w))+k(\delta) g(\delta w)=f \quad \text { on }\left(t_{0}, t_{f}\right),  \tag{6.4}\\
w\left(t_{0}\right)=w_{0},
\end{array}\right.
$$

where in (6.4): (i) Parameter $\delta$ is given in $\mathbf{R}$. (ii) $k(\cdot)$ is an increasing odd function of $\delta$ vanishing at 0 and Lipschitz continuous over $\mathbf{R}$. (iii)Function $g$ is of the following form:

$$
g(\xi)=\frac{a+b}{2}+\frac{b-a}{2}\left[\operatorname{sgn}(\xi)-\gamma_{g b}(\xi)\right]
$$

with $0<a<b$ and function $\gamma_{g b}$ of the same type than $\gamma$ (see Section 2 for details). The monotonicity, $\forall \delta \in \mathbf{R}$, of operator $w \rightarrow k(\delta) \operatorname{sgn}(\delta w)$, is the property making the above generalization possible.
6.2. Test problem with two degrees of freedom: We will describe in this section the numerical results obtained when applying the methodology of the previous
sections to a 2-degree of freedom model problem (2.1) (that is (2.4)). We take $T=4$ and

- the mass matrix $M=\left(\begin{array}{ll}2 & 1 \\ 1 & 2\end{array}\right)$, the stiffness matrix $A=\left(\begin{array}{rr}2 & -1 \\ -1 & 2\end{array}\right)$, the friction matrix $C=I$,
- $\gamma=\left\{\gamma_{i}\right\}_{i=1}^{2}$ with $\beta_{i}=\frac{1}{3}$ and $\varepsilon_{i}=10^{-1}, i=1,2$. (see Section 2),
- the external forcing term $f=\left\{f_{i}\right\}_{i=1}^{2}$, where

$$
f_{1}(t)= \begin{cases}2\left(t-\frac{t^{2}}{2}\right)-1-\gamma_{1}(1-t) & \text { if } 0 \leq t \leq 1 \\ 1+\left(t-\frac{3}{2}\right)-\gamma_{1}(0) & \text { if } 1 \leq t \leq 2 \\ 3 t^{3}-23 t^{2}+70 t-\frac{238}{3}-\gamma_{1}(4(t-3)(t-2)) & \text { if } 2 \leq t \leq 3 \\ \frac{t^{3}}{3}-3 t^{2}+6 t-\frac{17}{6}-\gamma_{1}(0) & \text { if } 3 \leq t \leq 4\end{cases}
$$

and

$$
f_{2}(t)= \begin{cases}\frac{t^{2}}{2}-2 t-\gamma_{2}(0) & \text { if } 0 \leq t \leq 1 \\ \frac{1}{2}-t-\gamma_{2}(0) & \text { if } 1 \leq t \leq 2 \\ -2 t^{3}+16 t^{2}-36 t+\frac{163}{6}-\gamma_{2}\left(1-(t-3)^{2}\right) & \text { if } 2 \leq t \leq 3 \\ \frac{-2}{3} t^{3}+6 t^{2}-20 t+\frac{175}{6}-\gamma_{2}\left(1-(t-3)^{2}\right) & \text { if } 3 \leq t \leq 4\end{cases}
$$

For the above data, the solution of problem (2.1) is given by

$$
\begin{gathered}
v_{1}(t)= \begin{cases}1-t & \text { if } 0 \leq t<1, \\
0 & \text { if } 1 \leq t<2, \\
4(t-3)(t-2) & \text { if } 2 \leq t<3, \\
0 & \text { if } 3 \leq t \leq 4,\end{cases} \\
v_{2}(t)= \begin{cases}0 & \text { if } 0 \leq t<2, \\
1-(t-3)^{2} & \text { if } 2 \leq t \leq 4,\end{cases}
\end{gathered}
$$

and

$$
\begin{gathered}
x_{1}(t)= \begin{cases}t-\frac{t^{2}}{2} & \text { if } 0 \leq t<1, \\
\frac{1}{2} & \text { if } 1 \leq t<2, \\
\frac{1}{2}+4\left[\frac{1}{3}\left(t^{3}-8\right)-\frac{5}{2}\left(t^{2}-4\right)+6(t-2)\right] & \text { if } 2 \leq t<3, \\
\frac{-1}{6} & \text { if } 3 \leq t \leq 4,\end{cases} \\
x_{2}(t)= \begin{cases}0 & \text { if } 0 \leq t<2, \\
(t-2)-\frac{1}{3}\left((t-3)^{3}+1\right) & \text { if } 2 \leq t \leq 4,\end{cases}
\end{gathered}
$$

while the corresponding function $\lambda$ is given by

$$
\lambda_{1}(t)= \begin{cases}1 & \text { if } 0<t<1 \\ t-\frac{3}{2} & \text { if } 1<t<2 \\ -1 & \text { if } 2<t<3 \\ \frac{-1}{2} & \text { if } 3<t<4\end{cases}
$$

and

$$
\lambda_{2}(t)= \begin{cases}1-t & \text { if } 0<t<2 \\ 1 & \text { if } 2<t<4\end{cases}
$$

To solve problem (2.1), we have used the splitting scheme (2.5) - (2.9). The subproblem (2.6) is solved via scheme (3.13), (3.14), while the subproblem (2.8) is solved via a classical finite difference centered scheme. The following results have been obtained with $\Delta t=0.003$. On Figs. $5-10$, we have shown the graphs of the approximation of $\dot{X}, X, \lambda$, respectively. On Figs. 11-13, we have shown, on a log-scale, the $L^{2}$-error, on $\dot{X}, X, \lambda$, as functions of $\Delta t$. Once again, we observe first order accuracy. We observe also that the computed discrete multipliers do not exhibit spurious oscillations, as it is the case with other discretization schemes.


Fig. 5. The computed $v_{1}(t)$


Fig. 7. The computed $x_{1}(t)$


Fig. 6. The computed $v_{2}(t)$


Fig. 8. The computed $x_{2}(t)$


Fig. 9. The computed $\lambda_{1}(t)$ Fig. 10. The computed $\lambda_{2}(t)$


Fig. 11.
$L^{2}$-error on $\dot{X}$ versus $\tau$


Fig. 12.
$L^{2}$-error on $X$ versus $\tau \quad L^{2}$-error on $\lambda$ versus $\tau$


Fig. 14. The computed $v_{1}(t)$


Fig. 17. The computed $x_{1}(t)$


Fig. 20. The computed $\lambda_{1}(t)$


Fig. 15. The computed $v_{2}(t)$


Fig. 18. The computed $x_{2}(t)$


Fig. 21. The computed $\lambda_{2}(t)$


Fig. 16. The computed $v_{3}(t)$


Fig. 19. The computed $x_{3}(t)$


Fig. 22. The computed $\lambda_{3}(t)$
6.3. Test problem with three degrees of freedom: We will describe in this section the numerical results obtained when applying the methodology of the previous sections to a 3-degree of freedom model problem (2.1) (that is (2.4)). We take $T=4$ and

- the mass matrix $M=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3\end{array}\right)$, the friction matrix $\mathrm{C}=10 \mathrm{I}$, the stiffness matrix $A=\mathbf{0}$ and $\gamma=\mathbf{0}$,
- the external forcing term $f=\left\{f_{i}\right\}_{i=1}^{3}$, where $f_{i}(t)=-20 e^{-4 t}$.

To solve problem (2.1), we have used the same schemes and steps as the second test problem. The following results have been obtained with $\Delta t=0.003$. On Figs. $14-22$, we have shown the graphs of the approximation of $X, X, \lambda$, respectively. Based on the previous two test problems, a first order accuracy is expected for this test problem as well. We observe also that the computed discrete multipliers do not exhibit spurious oscillations, as it is the case with other discretization schemes.

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