CORRECTION AND ADDITION

1. Correction

(1) Page 77, Line 11: change "a derivation" to "a derivation D".
(2) Page 77, Line 12: change "T" to "D".
(3) Page 77, Line 14: change "A derivation" to "A derivation D".
(4) Page 77, Line 15: change "an F-derivation" to "an F-derivation D".
(5) Page 82, Line -3: change "$\{0\}$" to "0".
(6) Page 87, Line 15: change "$\begin{pmatrix} 0 \\ \alpha_2 \\ 0 \\ 0 \end{pmatrix}$" to "$\begin{pmatrix} 0 \\ \alpha_2 \\ 0 \\ \alpha_4 \end{pmatrix}$"
(7) Page 91, Line 1, 3: change "strong order" to "strong unit".
(8) Page 92, Line -6: change "strong order" to "strong unit".
(9) Page 101, Line 2: change "I_0" to "$I_0(R)$".
(10) Page 125, Line -5: change "be the upper bound" to "be the set of upper bounds".
(11) Page 139, Line 10: change "weak unit" to "weak order unit". Same change on lines 15, 20, 22, 24, also on page 140, lines 4, 20, 24, and on page 141, lines 2, 5.
(12) Page 247, Line -3: change "weak unit" to "weak order unit".

2. Addition

(1) We provide some conditions for an ℓ-ring with squares positive being an f-ring.

Theorem 1. *Let R be an Archimedean ℓ-ring with squares positive. If R contains an f-element e such that $r(e) = \{0\}$ or $\ell(e) = \{0\}$, then R is an f-ring.*

Proof. Suppose that $a \wedge e = 0$. Then for any positive integer n, $(a - ne)^2 \geq 0$ implies that $nea \leq a^2 + n^2e^2$. Since $a \wedge e = 0$ and e is an f-element, $nea \wedge n^2e^2 = 0$. Thus $nea \leq a^2$, and hence $ea = 0$ since R is Archimedean. Therefore $a = 0$ since $r(e) = \{0\}$. Hence e is a weak order unit. Then by Corollary 4.6, R is an f-ring. □

(2) Directed partial orders on $\mathbb{C}$ and $\mathbb{H}$.

It is an open question if $\mathbb{C}$ can be made into an ℓ-field and if $\mathbb{H}$ can be made into a lattice-ordered division ring. The first question was posted by Birkhoff and Pierce in their 1956's paper, and it has been considered as one of the most important questions in the area of lattice-ordered rings. Recently Schwartz and Yang have proved that $\mathbb{C}$ can be made into a partially ordered field with a directed partial order. We assume that $\mathbb{R}$ is a partially ordered field with a directed partial order such that $n1 \leq q$ for some $q \in \mathbb{R}^+$ and all positive integers n, and $\mathbb{R}^+ \cap \mathbb{Q} = \mathbb{Q}^+$, where $\mathbb{Q}^+$ is the positive cone of the usual total order on $\mathbb{Q}$. The reader is referred to Schwartz and Yang's paper for details about those facts, and we will go from here to extend the partial order on $\mathbb{R}$ to $\mathbb{C}$ and $\mathbb{H}$.

For $a, b \in \mathbb{R}^+$, define $a \ll b$ if for all positive integers n, $na \leq b$. So we have $1 \ll q$, and for any $x \in \mathbb{R}^+$, there exists $y \in \mathbb{R}^+$ such that $x \ll y$. In fact, $1 \ll q$ implies that $x \ll xq = y$.

(I) First consider $\mathbb{C}$. Define positive cone $P = \{a + bi \mid a, b \in \mathbb{R}^+ \text{ with } b \ll a\}$. We first show that P is a partial order.

(a) $P \cap -P = \{0\}$ is clear.

(b) $P + P \subseteq P$. Suppose that $a + bi, c + di \in P$. Then $a, b, c, d \in R^+$ and $na \leq b, nc \leq d$ for all positive integers n. So $n(a + c) \leq (b + d)$ for all positive integers n, that is, $(a + bi) + (c + di) = (a + c) + (b + d)i \in P$.

(c) $PP \subseteq P$. Suppose that $a + bi, c + di \in P$. We show that $(a + bi)(c + di) = (ac - bd) + (ad + bc)i \in P$. Clearly $ac - bd, ad + bc \in \mathbb{R}^+$. For any positive integer n, $3nad + 3nbc + 3bd \leq ac + ac + ac = 3ac$ and $\frac{1}{3} \in \mathbb{R}^+$ imply that $nad + nbc + bd \leq ac$. Thus $n(ad + bc) \leq ac - bd$, that is, $ad + bc \ll ac - bd$. Hence $(ac - bd) + (ad + bc)i \in P$.

Therefore, $(\mathbb{C}, P)$ is a partially ordered ring with $P\mathbb{R} = \mathbb{R}^+$. Finally we show that the partial order P is directed. For $z = a + bi \in \mathbb{C}$ with $a, b \in \mathbb{R}$, since $\mathbb{R}^+$ is directed, $a = a_1 - a_2, b = b_1 - b_2$ with $a_1, a_2, b_1, b_2 \in \mathbb{R}^+$. Take $x \in \mathbb{R}^+$ such that $a_1 + b_1 + b_2 \ll x$. Then $x + b_1 i, (x - a) + b_2 i \in P$ and $z = (x + b_1 i) - ((x - a) + b_2 i)$. Hence P is directed.

(II) Next consider $\mathbb{H}$. Recall that $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ as a vector space over $\mathbb{R}$ with the multiplication as follows.

$$
\begin{aligned}
&(2.1) && && (a_0 + a_1 i + a_2 j + a_3 k)(b_0 + b_1 i + b_2 j + b_3 k) \\
&(2.2) && = && (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) + (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2)i \\
&(2.3) && && +(a_0 b_2 + a_2 b_0 + a_3 b_1 - a_1 b_3)j + (a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1)k.
\end{aligned}
$$

Now we define an element $a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ being positive if there are positive elements $a_1', a_1'', a_2', a_2'', a_3', a_3''$ in $\mathbb{R}$ such that

$$a_1 = a_1' - a_1'', a_2 = a_2' - a_2'', a_3 = a_3' - a_3'' \text{ and } a_1' + a_1'' + a_2' + a_2'' + a_3' + a_3'' \ll a_0.$$

We notice that $a_1' + a_1'' + a_2' + a_2'' + a_3' + a_3'' \ll a_0$ is equivalent to $a_1' \ll a_0, a_1'' \ll a_0, a_2' \ll a_0, a_2'' \ll a_0, a_3' \ll a_0$, and $a_3'' \ll a_0$.

Theorem 2. *Let P be the set of all positive elements of $\mathbb{H}$ defined above. Then P is the positive cone of a directed partial order on $\mathbb{H}$ with $P \cap \mathbb{R} = \mathbb{R}^+$.*

Proof. It is clear that $P + P \subseteq P$ and $P \cap -P = \{0\}$. We show that $PP \subseteq P$. Suppose that $a_0 + a_1 i + a_2 j + a_3 k, b_0 + b_1 i + b_2 j + b_3 k \in P$. Then

$$a_1 = a_1' - a_1'', a_2 = a_2' - a_2'', a_3 = a_3' - a_3'' \text{ and } a_1' + a_1'' + a_2' + a_2'' + a_3' + a_3'' \ll a_0,$$

and

$$b_1 = b_1' - b_1'', b_2 = b_2' - b_2'', b_3 = b_3' - b_3'' \text{ and } b_1' + b_1'' + b_2' + b_2'' + b_3' + b_3'' \ll b_0,$$

where $a_i', a_i'', b_i', b_i'' \in \mathbb{R}^+$, $i = 1, 2, 3$.

By direct calculations, we have

$$
\begin{aligned}
& a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2 \\
= \; & (a_0 b_1' + a_1' b_0 + a_2' b_3' + a_2'' b_3'' + a_3' b_2'' + a_3'' b_2') \\
& -(a_0 b_1'' + a_1'' b_0 + a_2' b_3'' + a_2'' b_3' + a_3' b_2' + a_3'' b_2''),
\end{aligned}
$$

and,

$$(a_0b_1' + a_1'b_0 + a_2'b_3' + a_2''b_3'' + a_3'b_2'' + a_3''b_2') \ll (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)$$
$$(a_0b_1'' + a_1''b_0 + a_2'b_3'' + a_2''b_3' + a_3'b_2' + a_3''b_2'') \ll (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3).$$

Similarly, coefficients of j and k in (3) are differences of two positive elements in $\mathbb{R}$ and each of those positive elements $\ll (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)$. Therefore the product of two elements in P is also in P, that is, $PP \subseteq P$.

Thus $(\mathbb{H}, P)$ is a partially ordered ring and it is clear that $P \cap \mathbb{R} = \mathbb{R}^+$.

We show that P is actually directed. Suppose that $a_0 + a_1i + a_2j + a_3k \in \mathbb{H}$ and $a_0 = a_0' - a_0'', a_1 = a_1' - a_1'', a_2 = a_2' - a_2'', a_3 = a_3' - a_3''$ with a_i', a_{ii}'' are positive in $\mathbb{R}$, $i = 0, 1, 2, 3$. Then

$$a_0 + a_1i + a_2j + a_3k = (a_0' + a_1'i + a_2'j + a_3'k) - (a_0'' + a_1''i + a_2''j + a_3''k).$$

Take $a \in \mathbb{R}^+$ such that $a_0' + a_1' + a_2' + a_3' \ll a$, we have

$$a_0' + a_1'i + a_2'j + a_3'k = (a + a_0') + a_1'i + a_2'j + a_3'k - a$$

with $(a + a_0') + a_1'i + a_2'j + a_3'k, a \in P$. Similarly $a_0'' + a_1''i + a_2''j + a_3''k$ is a difference of two elements in P. Hence $a_0 + a_1i + a_2j + a_3k$ is a difference of two elements in P. Therefore P is directed. This completes the proof. □

We notice that above method for constructing a directed partial order on $\mathbb{H}$ can be also used to produce a directed partial order on $\mathbb{C}$ with a larger positive cone than the positive cone defined on $\mathbb{C}$ before.

(3) Consider $\mathbb{R}$ with the total order. Birkhoff in his book "Lattice Theory, Colloquium Publications, AMS, **25**, third edition (1967)" asks how many ways that $\mathbb{H}$ can be made into a partially ordered algebra over $\mathbb{R}$ with a directed partial order. The following shows that $\mathbb{H}$ cannot be made into a partially ordered algebra over $\mathbb{R}$ with a directed partial order. This fact is true for any quaternions over a totally ordered subfield of $\mathbb{R}$.

Theorem 3. *$\mathbb{H}$ cannot be a directed algebra over $\mathbb{R}$ with the total order.*

Proof. Suppose that $\mathbb{H}$ is a directed algebra over $\mathbb{R}$. We derive a contradiction. We first show that for each $w = a_0 + a_1i + a_2j + a_3k > 0$ in $\mathbb{H}$, $a_0 \in \mathbb{R}^+$. In fact, since $w^2 - 2a_0w = -(a_0^2 + a_1^2 + a_2^2 + a_3^2)$ and for any $a \in \mathbb{R}^+, z > 0$ in $\mathbb{H}$, $az \geq 0$ in $\mathbb{H}$, if $a_0 \leq 0$ in $\mathbb{R}$, then $-a_0 \in \mathbb{R}^+$, and hence $w^2 - 2a_0w > 0$ in $\mathbb{H}$. Thus $-(a_0^2 + a_1^2 + a_2^2 + a_3^2) > 0$ in $\mathbb{H}$. Then $-(a_0^2 + a_1^2 + a_2^2 + a_3^2)w > 0$ in $\mathbb{H}$. On the other hand, $(a_0^2 + a_1^2 + a_2^2 + a_3^2) \in \mathbb{R}^+$ implies that $(a_0^2 + a_1^2 + a_2^2 + a_3^2)w > 0$ in $\mathbb{H}$. Therefore we obtain $(a_0^2 + a_1^2 + a_2^2 + a_3^2)w = 0$, which is a contradiction. Hence for each positive element in $\mathbb{H}$, its real part is positive in $\mathbb{R}$.

Since the partial order on $\mathbb{H}$ is directed, we can take $z = a + bi + cj + dk > 0$ in $\mathbb{H}$ with $z \notin \mathbb{R}$. By previous paragraph, $a \in \mathbb{R}^+$. If $a = 0$, then $z^2 = -(b^2 + c^2 + d^2) > 0$ in $\mathbb{H}$, which is contradiction. Thus $a > 0$ in $\mathbb{R}$. Since $\mathbb{R}$ is totally ordered, $a^{-1} > 0$ in $\mathbb{R}$, and hence $a^{-1}z = 1 + (a^{-1}b)i + (a^{-1}c)j + (a^{-1}d)k > 0$ in $\mathbb{H}$. Suppose that $a^{-1}b = s, a^{-1}c = t, a^{-1}d = u$. Then we have $w = 1 + si + tj + uk > 0$ in $\mathbb{H}$ and $w \notin \mathbb{R}$. For simplicity, let $v = si + tj + uk$. Then $v^2 = -(s^2 + t^2 + u^2)$, so $-v^2 \in \mathbb{R}^+$. We have

$$\begin{aligned} w = 1 + v \quad &\Rightarrow \quad w^2 = 1 + 2v + v^2 \\ &\Rightarrow \quad w^3 - v^2w = (1 + 2v)(1 + v) = 1 + 3v + 2v^2 > 0 \end{aligned}$$

in $\mathbb{H}$. Let $w_1 = 1 + 3v + 2v^2$. Then $w_1 > 0$ in $\mathbb{H}$, and hence $(w_1 - 2v^2)w = (1 + 3v)(1 + v) = 1 + 4v + 3v^2 > 0$ in $\mathbb{H}$. If we continue this procedure, we get that for any positive integer n, $(1 + nv)(1 + v) = 1 + (n + 1)v + nv^2 > 0$ in $\mathbb{H}$. Therefore since the real part of a positive element in $\mathbb{H}$ must be positive in $\mathbb{R}$, we must have $-nv^2 \leq 1$ for all positive integers n. Then $-v^2 = 0$, so $v^2 = 0$ since $\mathbb{R}$ is archimedean with respect to the total order. Hence $s = t = u = 0$, which is a contradiction.

Hence $\mathbb{H}$ cannot be a directed algebra over $\mathbb{R}$ with the total order. $\square$